Ironing, Sweeping, and Multivariate Majorization

Optimal Mechanisms for Mass-Produced Goods

(with Jacob Goeree (UNSW) and Ningyi Sun (UNSW))

Nick Bedard (WLU)

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Introduction

We consider monopoly that sells an excludable, non-rivalrous good

- For profit public goods
- Many mass-produced goods fit this framework
- E.g. newspapers, songs, movies, books, iPhones, television

Monopolist chooses single quality level to be enjoyed by *all* consumers

Monopolist can restrict access to the good

Buyers' valuations are interdependent: private and common values

- Could go either way: higher type for i may raise/lower j's value

Main Difficulty

The problem is naturally *irregular*

- i.e. incentive constraints cannot be substituted out

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Mussa and Rosen's (1978) and Myerson's (1981) "ironing"

- Constructive approach but only for unidimensional case

Rochet and Choné's (1998) "sweeping"

- Works for multidimensional case but not constructive

Main Difficulty

We develop a constructive multidimensional approach to ironing

- Extend majorization theory to higher dimensions
- Based on Kuhn-Tucker theory
- Implement ironing via simple quadratic minimization problems

Main Difficulty

Seller's problem:

$$\max_{q(\mathbf{x}) \text{ is non-decreasing}} \mathbb{E}\left[q(\mathbf{x}) MR(\mathbf{x}) - \tfrac{1}{2} q(\mathbf{x})^2 \right]$$

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Ironing: construct non-decreasing $\overline{MR}(\mathbf{x})$ such that

$$\tilde{q}(\mathbf{x}) = \arg\max_{q(\mathbf{x})} \mathbb{E}\left[q(\mathbf{x}) \overline{MR}(\mathbf{x}) - \frac{1}{2} q(\mathbf{x})^2 \right]$$

solves the original problem

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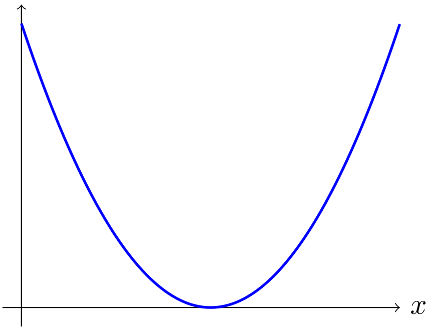
E.g.

$$\overbrace{\begin{pmatrix} 0 & 9 & 0 \\ 1 & 10 & 1 \\ 2 & 11 & 2 \end{pmatrix}}^{MR_1} + \overbrace{\begin{pmatrix} 0 & 1 & 2 \\ 9 & 10 & 11 \\ 0 & 1 & 2 \end{pmatrix}}^{MR_2} = \overbrace{\begin{pmatrix} 0 & 10 & 2 \\ 10 & 20 & 12 \\ 2 & 12 & 4 \end{pmatrix}}^{MR}$$

Main Difficulty



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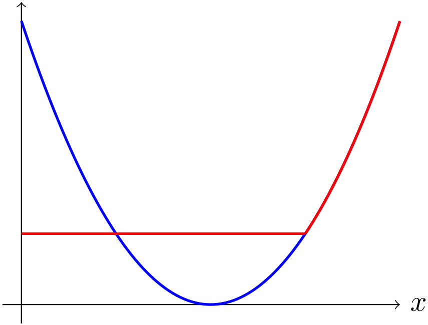

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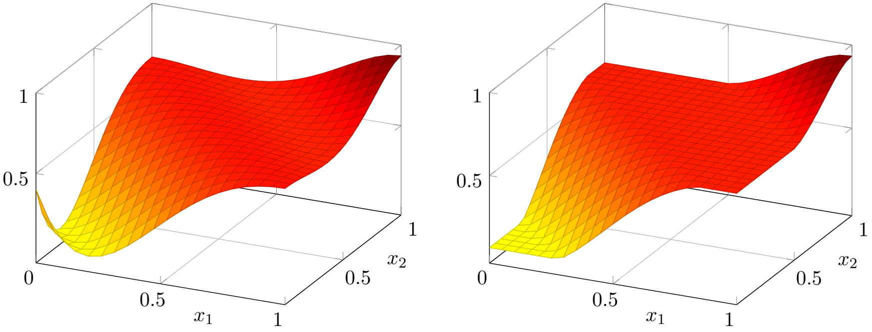

Model

Buyers $i = \{1, \dots, n\}$ with types x_i

- Types drawn independently according to distribution $F_i(x_i)$
- Highest type $\bar{x}_i$

Seller chooses

- quality: $q(\mathbf{x})$
- access rights: $\{\eta_1(\mathbf{x}), \dots, \eta_n(\mathbf{x})\}$
- transfers: $\{t_1(\mathbf{x}), \dots, t_n(\mathbf{x})\}$

Buyer i's utility: $u_i(\mathbf{x}) = v_i(\mathbf{x})q(\mathbf{x})\eta_i(\mathbf{x}) - t_i(\mathbf{x})$

Assume $v_i(x_i, \mathbf{x}_{-i})$ increasing in x_i for all $\boldsymbol{x}_{-i} \in X_{-i}$

Seller's Problem

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subject to (ex post) incentive compatibility: for all $i \in N$, $\mathbf{x} \in X$

$$u_i(\mathbf{x}) \in \arg\max_{\hat{x}_i \in X_i} u_i(\hat{x}_i, \mathbf{x}_{-i})$$

and individual rationality: for all $i \in N$, $\mathbf{x} \in X$

$$u_i(\mathbf{x}) \geq 0$$

Seller's Problem

Proposition

Mechanism $(q, \boldsymbol{\eta}, \boldsymbol{t})$ *is incentive compatible if and only if, for all* $i \in N$*,* $\mathbf{x} \in X$*,* $q(\mathbf{x})\eta_i(\mathbf{x})$ *non-decreasing in* x_i *and*

$$t_i(\mathbf{x}) = v_i(\mathbf{x})q(\mathbf{x})\eta_i(\mathbf{x}) - \sum_{s_i < x_i} \overline{\Delta}_i v_i(s_i, \boldsymbol{x}_{-i})q(s_i, \boldsymbol{x}_{-i})\eta_i(s_i, \boldsymbol{x}_{-i})$$

- $\overline{\Delta}_i v_i(s_i, \boldsymbol{x}_{-i}) = v_i(s_i^+, \boldsymbol{x}_{-i}) - v_i(s_i, \boldsymbol{x}_{-i})$ and s_i^+ is one type higher than s_i

Seller's Problem

Seller's problem can be written as

$$\max_{\substack{(q,\boldsymbol{\eta}):\mathbf{X}\to\mathbb{R}_{\geq 0}\times[0,1]^n \\ q(\mathbf{x})\eta_{\mathbf{i}}(\mathbf{x}) \text{ non-decreasing for all } i}} \mathbb{E}\Big[q(\mathbf{x})\sum_{i\in N} MR_i(\mathbf{x})\eta_i(\mathbf{x}) - C\big(q(\mathbf{x})\big)\Big]$$

where $MR_i(\mathbf{x}) = v_i(\mathbf{x}) - \overline{\Delta}_i v_i(\mathbf{x})\frac{1-F_i(x_i)}{f_i(x_i)}$

Seller's Problem: Public Access and Quadratic Costs

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Seller's Problem: Public Access

Saddle-point problem

$$\min_{\boldsymbol{\lambda}: X \to \mathbb{R}_{\geq 0}} \max_{q: X \to \mathbb{R}_{\geq 0}} \left\{ \mathbb{E}\left[q(\mathbf{x}) \sum_{i \in N} MR_i(\mathbf{x}) - \frac{1}{2} q(\mathbf{x})^2 \right] + \sum_{\substack{i \in N \\ \mathbf{x} \in X}} \lambda_i(\mathbf{x}) \overline{\Delta}_i q(\mathbf{x}) \right\}$$

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where $\lambda_i(\bar{x}_i, \mathbf{x}_{-i}) = 0$ and $\underline{\Delta}_i v_i(s_i, \boldsymbol{x}_{-i}) = v_i(s_i, \boldsymbol{x}_{-i}) - v_i(s_i^-, \boldsymbol{x}_{-i})$ and s_i^- is one type higher than s_i.

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Univariate Majorization

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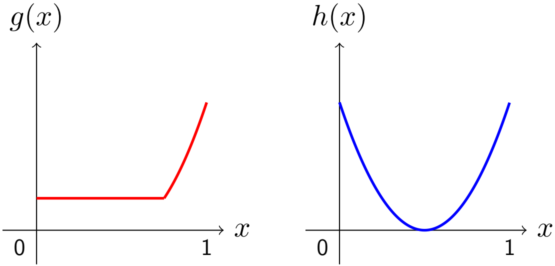
Univariate Majorization



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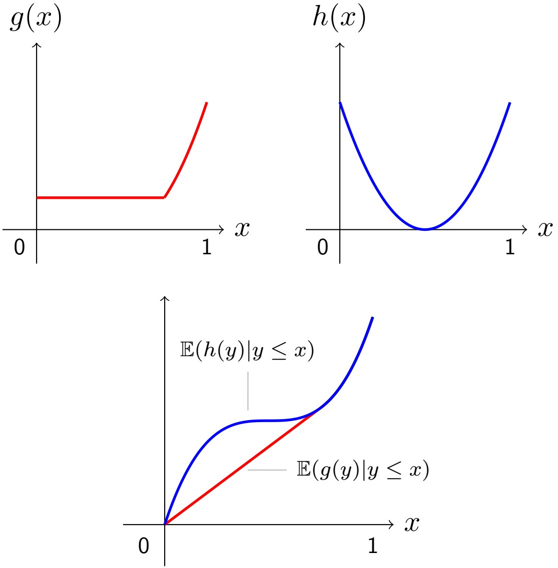

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$$= \mathbb{E}\Big[MR(s, \mathbf{x}_{-i}) \,|\, s \leq x_i\Big] - \sum_{s \leq x_i} \underline{\Delta}_i \lambda_i(s, \mathbf{x}_{-i})$$

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&= \mathbb{E}\Big[MR(s, \mathbf{x}_{-i}) \,\Big|\, s \leq x_i\Big] - \lambda_i(x_i, \mathbf{x}_{-i}) \\
&\leq \mathbb{E}\Big[MR(s, \mathbf{x}_{-i}) \,\Big|\, s \leq x_i\Big]
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Multivariate Majorization

(i) $\sum_{i \in N}\left(MR_i(\mathbf{x}) - \frac{\underline{\Delta}_i \lambda_i(\mathbf{x})}{f_i(x_i)}\right)$ has lower lower-sums than $\sum_{i \in N} MR_i(\mathbf{x})$: for all lower sets $X_- \subset X$

$$\mathbb{E}\Big[\sum_{i \in N}\Big(MR_i(\mathbf{x}) - \frac{\underline{\Delta}_i \lambda_i(\mathbf{x})}{f_i(x_i)}\Big) \,|\, \mathbf{x} \in X_-\Big] \leq \mathbb{E}\Big[\sum_{i \in N} MR_i(\mathbf{x}) \,|\, \mathbf{x} \in X_-\Big]$$

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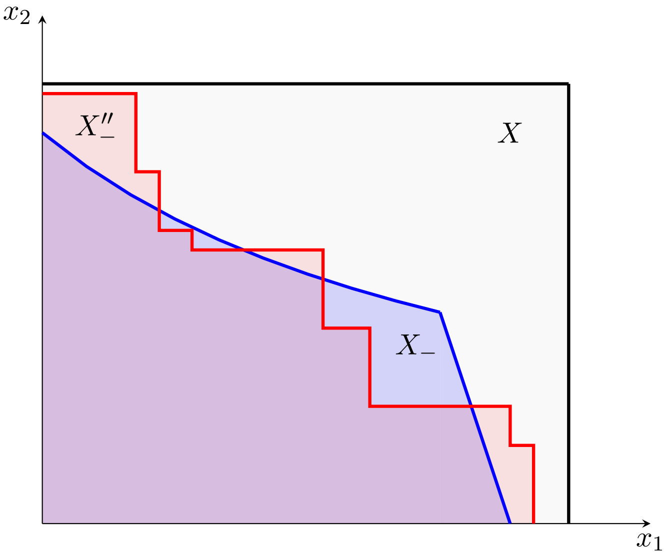
Multivariate Majorization

Lower Sets



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Seller's Problem: Public Access

Saddle-point problem

$$\min_{\boldsymbol{\lambda}:X\to\mathbb{R}_{\geq 0}} \max_{q:X\to\mathbb{R}_{\geq 0}} \mathbb{E}\Big[q(\mathbf{x})\sum_{i\in N}\Big(MR_i(\mathbf{x}) - \frac{\underline{\Delta}_i\lambda_i(\mathbf{x})}{f_i(x_i)}\Big) - \frac{1}{2}q(\mathbf{x})^2\Big]$$

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Saddle-point problem

$$\min_{\boldsymbol{\lambda} : X \to \mathbb{R}_{\geq 0}} \tfrac{1}{2}\mathbb{E}\left[\max\left\{0, MR(\mathbf{x}) - \underline{\Delta}_i \lambda_i(\mathbf{x})\right\}^2\right]$$

Seller's Problem: Public Access

Saddle-point problem

$$\min_{\substack{g\,:\,X\to\mathbb{R} \\ g\succ MR}} \tfrac{1}{2}\mathbb{E}[\max\{0,g(\mathbf{x})\}^2]$$

where $MR = \sum_{i\in N} MR_i$

Ironing Independent of C

Lemma

If

$$\overline{MR}(\mathbf{x}) = \underset{\substack{g : X \to \mathbb{R} \\ g \succ MR}}{\arg\min}\, \mathbb{E}[g(\mathbf{x})^2]$$

Ironing Independent of C

Lemma

If

$$\overline{MR}(\mathbf{x}) = \operatorname*{arg\,min}_{\substack{g\,:\,X \to \mathbb{R} \\ g \succ MR}} \mathbb{E}[g(\mathbf{x})^2]$$

then

$$\overline{MR}(\mathbf{x}) \;=\; \operatorname*{arg\,min}_{\substack{g\,:\,X \to \mathbb{R} \\ g \succ MR}} \mathbb{E}\big[\psi(g(\mathbf{x}))\big]$$

for any convex function $\psi : \mathbb{R} \to \mathbb{R}$*.*

Seller's Problem: Public Access

For any convex C, ironed MR are solution to the simple quadratic problem:

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- $\overline{MR}(\mathbf{x})$ *is non-decreasing in all directions*
- $\overline{MR}(\mathbf{x})$ *is minimal in* $\{g \,|\, g \succ MR\}$ *with respect to* $\succ$*:*

$$\overline{MR} \succ g \succ MR \text{ implies } g = \overline{MR}$$

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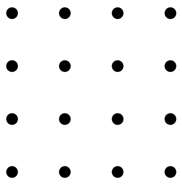

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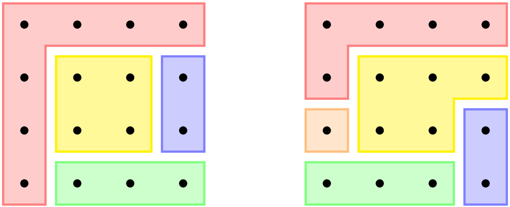

Seller's Problem: Public Access

For any convex C, ironed MR are solution to the simple quadratic problem:

$$\overline{MR}(\mathbf{x}) = \underset{\substack{g : X \to \mathbb{R} \\ g \succ MR}}{\arg\min} \, \mathbb{E}[g(\mathbf{x})^2]$$

Proposition

- *Level sets of $\overline{MR}(\mathbf{x})$ form an ortho-convex partition*
- *For each cell P of the partition $\overline{MR}(\mathbf{x}) = \mathbb{E}[MR(\mathbf{y})|\mathbf{y} \in P]$*

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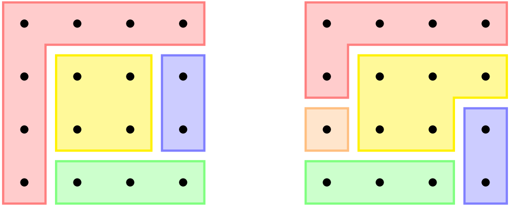

Example: Public Access

Suppose

- 2 buyers
- types uniform on $X_1 = X_2 = \{0, 1, 2\}$,
- valuations $v_i(x_i, x_{-i}) = 1 + \frac{1}{2}x_i + 9x_{-i}(2 - x_{-i})$ for $i = 1, 2$

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$$MR = \begin{pmatrix} 0 & 10 & 2 \\ 10 & 20 & 12 \\ 2 & 12 & 4 \end{pmatrix}$$

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$$MR = \begin{pmatrix} 0 & 10 & 2 \\ 10 & 20 & 12 \\ 2 & 12 & 4 \end{pmatrix} \longrightarrow \overline{MR} = \begin{pmatrix} 0 & 6 & 6 \\ 6 & 12 & 12 \\ 6 & 12 & 12 \end{pmatrix}$$

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$$-\underline{\Delta}_1\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ -4 & -4 & -4 \\ 4 & 4 & 4 \end{pmatrix}, \quad -\underline{\Delta}_2\lambda_2 = \begin{pmatrix} 0 & -4 & 4 \\ 0 & -4 & 4 \\ 0 & -4 & 4 \end{pmatrix}$$

Optimal Mechanism: Public Access

Proposition

The optimal mechanism for public goods is given by $\{q^*(\mathbf{x}), t_i^*(\mathbf{x})\}$ *where*

$$q^*(\mathbf{x}) \;=\; C'^{-1}\big(\max(0, \overline{MR}(\mathbf{x}))\big)$$

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$$t_i^*(\mathbf{x}) \;=\; v_i(\mathbf{x})q^*(\mathbf{x}) - \sum_{s_i < x_i} \overline{\Delta} v_i(s_i, \mathbf{x}_{-i})q^*(s_i, \mathbf{x}_{-i})$$

How can restricted access can help seller?

Suppose

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With public access, we had

$$MR = \begin{pmatrix} 0 & 10 & 2 \\ \boxed{10} & 20 & 12 \\ \boxed{2} & 12 & 4 \end{pmatrix} \longrightarrow \overline{MR} = \begin{pmatrix} 0 & 6 & 6 \\ \boxed{6} & 12 & 12 \\ \boxed{6} & 12 & 12 \end{pmatrix}$$

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Individual marginal revenue:

$$MR_1 = \begin{pmatrix} 0 & 9 & 0 \\ 1 & 10 & 1 \\ 2 & 11 & 2 \end{pmatrix}, MR_2 = \begin{pmatrix} 0 & 1 & 2 \\ 9 & 10 & 11 \\ 0 & 1 & 2 \end{pmatrix}$$

Seller's Problem

Restricted Access

Saddle-point problem

$$\min_{\lambda : X \to \mathbb{R}_+} \max_{(q,\boldsymbol{\eta}) : X \to \mathbb{R}_{\geq 0} \times [0,1]^n} \mathbb{E}\Big[q(\mathbf{x}) \sum_{i \in N} \eta_i(\mathbf{x}) \Big(MR_i(\mathbf{x}) - \frac{\underline{\Delta}_i \lambda_i(\mathbf{x})}{f(\mathbf{x})} \Big) - \frac{1}{2} q(\mathbf{x})^2 \Big]$$

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Restricted Access

Saddle-point problem

$$\min_{\lambda : X \to \mathbb{R}_+} \max_{q : X \to \mathbb{R}_{\geq 0}} \mathbb{E}\left[q(\mathbf{x}) \sum_{i \in N} \max\left\{ 0, MR_i(\mathbf{x}) - \frac{\underline{\Delta}_i \lambda_i(\mathbf{x})}{f(\mathbf{x})} \right\} - \frac{1}{2} q(\mathbf{x})^2 \right]$$

Seller's Problem

Restricted Access

Saddle-point problem

$$\frac{1}{2} \max_{\substack{g_i : X \to \mathbb{R} \\ g_i \succ MR_i}} \mathbb{E}\Big[\Big(\sum_{i \in N} \max(0, g_i(\mathbf{x})\Big)^2\Big]$$

Restricted Acccess

Proposition

The optimal mechanism for excludable goods is given by $(q^*, \boldsymbol{\eta}^*, \mathbf{t}^*)$ *with*

$$q^*(\mathbf{x}) = C'^{-1}\Big(\sum_{i \in N} \max\big(0, \widetilde{MR}_i(\mathbf{x})\big)\Big)$$

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... and, for $i \in N$*,*

$$\eta_i^*(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad \widetilde{MR}_i(\mathbf{x}) < 0 \\ \eta_i^0(\mathbf{x}) & \text{if} \quad \widetilde{MR}_i(\mathbf{x}) = 0 \\ 1 & \text{if} \quad \widetilde{MR}_i(\mathbf{x}) > 0 \end{cases}$$

where $\eta_i^0(\mathbf{x})$ *is such that* $q^*(\mathbf{x})\eta_i^0(\mathbf{x})$ *is constant in* x_i *on each cell of the generated partition and non-decreasing in* x_i*,*

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Need $q_1^*(1,0)\eta_1^*(1,0) = q_1^*(2,0)\eta_1(2,0)$

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Need $q_1^*(1,0)\eta_1^*(1,0) = q_1^*(2,0)\eta_1(2,0) \Leftrightarrow \eta_1^*(1,0) = \frac{1}{3}$

$$\eta_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & 1 & \frac{3}{7} \\ 1 & 1 & 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ 1 & 1 & 1 \\ 0 & \frac{3}{7} & 1 \end{pmatrix}$$

Conclusions

We solve problem of a monopolist provider of non-rivalrous good

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Future work:

- Univariate majorization is related to 2nd order stochastic dominance
- Multivariate majorization gives natural generalization to multivariate case