Optimal dynamic insurance contracts*

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Abstract

I analyze long-term contracting in insurance markets with asymmetric information. A buyer privately observes her risk type, which evolves stochastically over time. A long-term contract specifies a menu of policies contingent on the history of coverage choices and contractable accident information. The optimal contract offers in each period a choice between a perpetual complete coverage policy with fixed premium, and a risky continuation contract in which accidents affect within-period consumption (partial coverage) and, potentially, future policies.

I allow for restrictions on how accident information can be used in pricing. Without restrictions, accidents and coverage choices are used as signals for the efficient provision of incentives. In the presence of pricing restrictions, low coverage is rewarded, leading to menus with more attractive policies. Allocative inefficiency decreases along all histories. These results are used to study a model of perfect competition, where the equilibrium is unique whenever it exists, and the monopoly problem.

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1 Introduction

Most insurance contracts cover risks that are present over multiple periods, such as health, auto or home insurance. This allows insurance firms to benefit from dynamic pricing schemes, where consumers' premium and coverage evolve over time and incorporate information observed over the course of their interaction. Coverage and premium dynamics are a central issue in several insurance markets including auto (Dionne and Doherty (1994), Cohen (2005)), health (Handel et al. (2015), Atal (2019), Atal et al. (2020)) and life (Hendel and Lizzeri (2003), Daily et al. (2008)). This raises the fundamental question of what dynamic pricing schemes arise as a result of profit maximization and what are their consequences for coverage and premium dynamics. The theoretical insurance literature has focused on scenarios with either symmetric risk information or permanent risk types. In this paper, I characterize profit maximizing long-term contracts in repeated interactions with evolving, persistent and private risk information.

In my model, a risk-averse consumer (she) may incur incidental losses in each period, such as a car accident (auto insurance) or medical expenses (health insurance); and the probability distribution over losses is determined by her risk-type, which is privately known and follows a persistent Markov process. A high (low) type has a higher (lower) expected expected income, net of losses. An insurance firm offers a long-term contract to the consumer after she has observed her initial risk-type. A long-term contract represents a commitment to a schedule in which a menu of insurance policies (referred to as flow contracts) with different premiums and coverage levels is offered to the consumer in each period. The menu offered on a given period may depend on previous choices made by the consumer as well as any available information regarding accident history. Motivated by regulatory policies that limit the use of accident history in pricing (Handel et al. (2015), Farinha Luz et al. (2021)), I allow for restrictions on the amount of information about past accidents that can be explicitly used by firms in setting insurance policy offers to the consumer. These restrictions include as special cases both fully contingent contracts, where the whole history of accidents can be used, and realization-independent contracts, where no past accident information is used.

The first part of the paper (Sections 3-6) considers fixed discounted utility levels for the consumer, dependent on her initial risk-type, and characterizes the profit maximizing long-term contract that delivers these utility levels. This allows for the derivation of qualitative properties of optimal contracts that hold in both the competitive (Section 7) and monopolistic (Section 8) settings studied in the second part of the paper.

The optimal contract features a simple pricing scheme: in every period the consumer chooses between a complete coverage policy with a perpetual fixed premium (efficient in this model), and a partial coverage policy, in which case future offers depend on additional accident realizations. The consumer is induced to choose full coverage when having a low-type realization, and to choose partial coverage when having successive high-type realizations.

In order to efficiently screen different risk types, policy offers in the optimal mechanism incorporate two sources of information: accident information — which is directly observed by the firm — and the consumer's choice (or announcement in a direct mechanism). Accident information that is indicative of high-types is rewarded with higher continuation utility, in the form of more attractive future contracts. Similarly, the choice of partial coverage in a given period is indicative of a high-type and is rewarded with higher continuation utility.

The study of distortion and coverage dynamics is challenging due to two technical challenges. First, flow contracts are multi-dimensional objects, describing a premium and coverage for each possible loss level. Second, the presence of risk aversion leads to a non-separability issue: the separation of consumers with different types requires the introduction of distortions, in the form of partial coverage, and the marginal efficiency loss from partial coverage potentially depends on the the consumer's utility level in a given period. As a consequence, the optimal contract jointly optimizes over the intertemporal allocation of utility and distortions. I tackle these issues by introducing an auxiliary static cost minimization problem in Section 5, which finds the optimal flow contract that (i) delivers a certain utility level if consumed by a high-type, and (ii) provides a fixed punishment to a low-type pretending to be high-type. In Section 6, I characterize both the efficient spreading of utility and distortions over time, in terms of this auxiliary cost function. Under a mild condition on the consumer's preferences, the auxiliary cost function is supermodular, in which case I obtain a sharp characterization of the dynamics of utility and distortions.

For the case of realization-independent contracts, distortions — consumer's exposure to risk — in the optimal mechanism are decreasing along *all* type-paths. When it comes to the flow utility dynamics, I show that consecutive high-type announcements — which are revealed by the choice of partial coverage — are rewarded by leading to an offer of partial coverage generating higher flow utility, as well as lower distortion. With fully contingent contracts, I present results for two special cases. First, in the case of two periods I show that the firm has an incentive to reward partial coverage and reduce distortions over time. I recast the firm's contract design problem as a cost minimization one and show that, when averaging over possible first-period income realizations, the expected marginal cost of distortions in flow contracts decreases over time, while the expected marginal cost of flow utility increases over time following the choice of partial coverage. Second, on the illuminating knife-edge case of utility $u(c) = \sqrt{c}$, which corresponds to a separable auxiliary cost function, and an arbitrary number of periods, consecutive choices of partial coverage — which corresponds to high-type announcements — lead to a path of insurance policies with lower distortions and increasing flow utility levels over time, when averaging over possible income realizations.

In Section 7, I consider a competitive model in which multiple firms make long-term contract offers to a consumer who is able to commit to a long-term contract. I extend the characterization of Rothschild and Stiglitz (1976) and show that a unique outcome, featuring the flow utility and distortion dynamics aforementioned, can be sustained by a pure strategy equilibrium, and obtain necessary and sufficient conditions for such an equilibrium to exist.

The assumption of consumer commitment is reasonable in markets with high search or switching costs which inhibit the consumers' transition to new firms. In the absence of such frictions, the commitment assumption is potentially with loss. If the low-type is an absorbing state of the types' Markov process (such as a chronic condition in health insurance), I present a model with consumer reentry in the market and show that firms' endogenous beliefs about the type of consumer searching for a new contract discourages consumer's firm switching and, as a consequence, the commitment outcome can be sustained in a Perfect Bayesian Equilibrium. This follows from the fact that the optimal contract with commitment is back-loaded: both the partial- and the full-coverage policy options within the optimal contract become more attractive over time.

Section 8 studies the case of monopoly and shows that the optimal contract is always separating and hence features the flow utility and distortion dynamics aforementioned. The consumer with initial high-type has no information rent. I present a condition that is necessary and sufficient for the optimal contract to leave information rents to the consumer with initial low-type, meaning that her utility is strictly higher than the no-insurance option.

Related literature

This paper contributes to the literature on competitive screening. The seminal contributions of Rothschild and Stiglitz (1976) and Wilson (1977) study a static competitive insurance model with private risk information and show that partial coverage serves as a screening device. Cooper and Hayes (1987) extends their analysis to a multi-period setting in which consumers have fixed risk types and full commitment and show that optimal long-term contracts use experience rating as an efficient screening device. Different from my paper, their optimal contract features a single type announcement in the first period. Dionne and Doherty (1994) allows for renegotiation in the same fixed-types model and find equilibria with semi-pooling.

My analysis is also related to the dynamic mechanism design literature (see Courty and Li (2000), Pavan et al. (2014)), which differs from my model in two ways. First, it assumes that the only source of information during interactions is the series of announcements by the consumer, while incomes are an additional information source in my model. Second, they focus on quasi-linear environments where transfers are only pinned down up to their total ex-post present value while the optimal contract is unique in my environment.

A notable exception is Garrett and Pavan (2015), which studies managerial compensation in the presence of persistent productivity private information in a two-period model. It study the dynamics of distortions (effort under-provision) by constructing two sets of perturbations that retain implementability. With perfect type persistence, the implementation of constant effort requires additional consumption variation in the second period, which is costly for a risk averse manager and as a result distortions are increasing in the optimal mechanism. The reverse occurs if persistence and risk aversion are low. While consumption dispersion is necessary for efficient effort in their model, it is the source of distortions in my framework. Battaglini (2005) considers the design of dynamic selling mechanisms in a monopoly setting with quasi-linearity where a consumer's valuation follows a binary Markov chain. Production in the optimal mechanism becomes efficient when the customer has a high value and converges to efficiency following consecutive low values, similar to my result for realization-independent contracts.

Hendel and Lizzeri (2003) studies life insurance contracts with symmetric information and one-side commitment and show that optimal contracts feature front-loaded payments, which serves as a lock-in device. Using U.S. data, it shows that front-loading is a common feature of contracts and is negatively correlated with net present value of premiums. Ghili et al. (2021) characterizes the optimal dynamic contract in health insurance assuming symmetric risk information and one-sided commitment, which features full insurance in each period together with a consumption floor, which is adjusted upwards whenever the consumer's participation constraint becomes binding. It estimates a model of health status and medical expenses using data from Utah and numerically calculates the optimal long-term contract.

2 Model

A consumer (she) lives for $T \leq \infty$ periods. At the start of each period, she privately observes her type $\theta_t \in \Theta \equiv \{l, h\}$ (low or high), which determines the probability distribution of realized income $y_t \in Y$, with $Y \subset \mathbb{R}_+$ finite. The occurrence of higher losses is represented by a lower level of income. In each period t, the probability distribution of income y_t is $p_{\theta_t} \in \Delta Y$, satisfying $\inf_{y \in Y} p_{\theta_t}(y) > 0$ and

$$\sum_{y \in Y} p_l(y)\, y < \sum_{y \in Y} p_h(y)\, y.$$

I assume types $\{\theta_t\}_{t=1}^{T}$ follow a Markov process with transition probabilities $\pi_{ij} \equiv \mathbb{P}(\theta_{t+1} = j \mid \theta_t = i)$, and the distribution of initial type θ_1 is denoted by $\pi_i \equiv \mathbb{P}(\theta_1 = i)$. Types are persistent, i.e., having a given type in period $t < T$ leads to a higher probability of having the same type in period $t + 1$. In short, I assume that $\pi_{ii} > \pi_{ji}$, for $i, j \in \{l, h\}$.

Consumer preferences over final consumption flows are determined by Bernoulli utility function $u : \mathbb{R}_+ \mapsto \mathbb{R}$,, assumed to be in C^2, strictly concave, strictly increasing with inverse $\psi(\cdot)$. The consumer has discount factor $\delta \in (0,1)$. Her utility from deterministic consumption stream $\{c_t\}_{t=1}^T$ is given by

$$\sum_{t=1}^{T} \delta^{t-1} u(c_t).$$

A risk-neutral firm has total profits, given realization of the income and consumption paths $\{y_t, c_t\}_{t=1}^T$, given by the discounted sum (using δ) of its net payments $\{c_t - y_t\}_{t=1}^T$ made to the consumer. The firm and consumer maximize discounted expected payoffs.

A flow contract is a single period insurance policy specifying a premium and coverage for all possible income realizations. Income realizations are observable and contractable and hence the execution of a flow contract is frictionless. I describe a flow contract through the induced final consumption of the consumer, which is her income plus any policy coverage payments minus the premium paid, i.e., the set of flow contracts is $Z \equiv \mathbb{R}_+^Y$. If a flow contract z is provided to a consumer with type θ, the consumer's utility and the profits obtained by a firm can be described, respectively, by

$$v(z,\theta) \equiv \sum_{y \in Y} p_\theta(y)\, u[z(y)] \text{ and } \xi(z,\theta) \equiv \sum_{y \in Y} p_\theta(y)\, [y - z(y)].$$

A long-term contract is a mechanism which specifies, at the initial period, the menu of flow contracts to be offered at each period, which may depend on the consumer's previous choices and income realizations. In several markets, regulatory restrictions limit the extent to which firms can explicitly use one's history of losses in pricing (see Handel et al. (2015), for example). These restrictions are modelled through a *signal structure*, which is composed of a finite *set of signals* Φ and a surjective *signal function* $\phi : Y \mapsto \Phi$. The signal structure is exogenously fixed. It imposes restrictions on the firm's contract design problem, as the income realization y_t can only impact future offers via $\phi_t \equiv \phi(y_t)$. A special case is that of *fully-contingent mechanisms*, with $\Phi = Y$ and $\phi(y) = y$ for all $y \in Y$, where prices may depend explicitly on both the consumer's history of choices and accidents. Another special case is that of *realization independent mechanisms*, where Φ is a singleton. In this case, firms are unable to use the history of previous income realizations in pricing.

From the revelation principle, we can restrict attention to direct mechanisms, where the consumer makes a type announcement in each period, and truthful equilibria, in which the consumer finds it optimal to truthfully report her type. I denote the history of announcements and signals up to period t as the observable history $\eta^t \in H^t \equiv \Phi^t \times \Theta^t$, with $H^0 \equiv \{\emptyset\}$.

I also denote a history of types $(\theta_1, \ldots, \theta_t)$ up to period t as θ^t, the expanded history $(\theta_1, \ldots, \theta_t, \theta_{t+1})$ as (θ^t, θ_{t+1}), the sub-history $(\theta_\tau, \ldots, \theta_{\tau'})$, for $\tau \leq \tau' \leq t$, as $[\theta^t]_\tau^{\tau'}$ and, for a history θ^t and $\tau \leq t$, refer to θ_τ as $[\theta^t]_\tau$. The same notation is used for income realizations y_t, signal realizations ϕ_t and histories η^t. Also define h^t as the t-period history $(h, \ldots, h)$.

A direct mechanism (referred to as a mechanism henceforth) is defined as $M = \{z_t\}_{t=1}^T$, with $z_t : H^{t-1} \times \Theta \mapsto Z$, and the set of such mechanisms is $\mathcal{M}$. A mechanism $M = \{z_t\}_{t=1}^T$ specifies the flow contract $z_t\left(\eta^t, \hat{\theta}_t\right)$ to be provided to the consumer at each period t, which depends on the history of signals and announcements up to period $t-1$, as well as the report $\hat{\theta}_t$. The flow contract at period t determines the level of coverage obtained by the consumer within period t, with her realized consumption in period t given by $z_t\left(y_t \mid \eta^t, \hat{\theta}_t\right)$. Figure 1 illustrates the sequence of events within a mechanism. Regardless of the signal structure, flow contracts with arbitrary income-contingent transfers can be executed without frictions. In the example of health insurance, mechanisms can specify and enforce transfers that depends directly on health shocks within a given period, but may — depending on ϕ — not be able to use this information to set future offers to be made to the consumer.

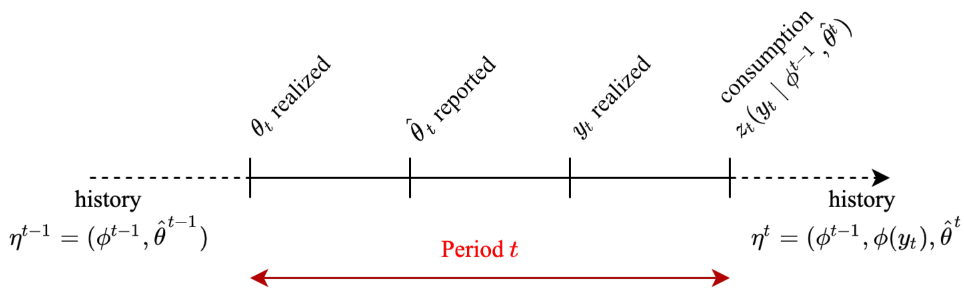

Figure 1: Timing of events of a particular period t within a mechanism

A private history in period t includes all information available to the consumer, and is described as $\eta_p^t = \left(y^t, \hat{\theta}^t, \theta^t\right) \in H_p^t$, which includes the history of income realizations y^t, type realizations θ^t and reported types $\hat{\theta}^t$. A reporting strategy is denoted by $r = \{r_t\}_{t=1}^T \in R$ with $r_t : H_p^{t-1} \times \Theta \mapsto \Theta$. The truth-telling strategy is denoted by r^* and satisfies $r_t^*\left(\eta_p^{t-1}, \theta\right) = \theta$, for all $\eta_p^{t-1} \in H_p^{t-1}$ and $\theta \in \Theta$. The history of reports up to period t generated on-path by strategy r is denoted as $r^t\left(y^{t-1}, \theta^t\right)$. The consumer's payoff from a reporting strategy $r \in R$ is denoted as

$$V_0^r(M) \equiv \mathbb{E}\left\{\sum_{t=1}^T \delta^{t-1} v\left[z_t\left(\phi^{t-1}, r^t\left(y^{t-1}, \theta^t\right)\right), \theta_t\right]\right\}.$$

A mechanism M is incentive compatible if $V_0^{r^*}(M) \geq V_0^r(M)$ for all $r \in R$. The set of incentive compatible mechanisms is denoted as $\mathcal{M}_{IC}$. Finally, for an observable history $\eta^{t-1} = (\theta^{t-1}, y^{t-1}) \in H^{t-1}$ and period t type $\theta_t \in \Theta$, I denote the continuation utility of a consumer following reporting strategy r as $V_t^r\left(M \mid \eta^{t-1}, \theta_t\right)$, or simply as $V_t\left(M \mid \eta^{t-1}, \theta_t\right)$ for reporting strategy r^*. If the referred mechanism is clear, I use $V_t(\eta^{t-1}, \theta_t)$ for brevity.

3 Profit maximization

I assume throughout that the consumer privately learns θ_1 prior to contracting with the firm. A firm's mechanism design problem can be separated into two parts. First, a firm must decide on how attractive its offer is for different types of potential customers, which is described by the total expected utility from participation (utility choice). Second, firms must choose the mechanism in order to maximize profits within all mechanisms that deliver the same utility to the consumer (feature design). I start by focusing on the feature design problem and provide a characterization of profit maximizing mechanisms for fixed discounted expected utility to be provided to the consumer with each initial type. The study of the feature design problem allows for the characterization of qualitative properties of optimal mechanisms that hold under multiple market structures which lead to different equilibrium utility levels for the consumer. Sections 7 and 8 analyze the cases of perfect competition and monopoly, respectively. The total discounted expected profits from a mechanism $M \in \mathcal{M}$ is given by

$$\Pi\left(M\right) \equiv \mathbb{E}\left[\sum_{t=1}^{T} \delta^{t-1} \xi\left(z_t\left(\eta^{t-1}, \theta_t\right), \theta_t\right)\right]. \tag{1}$$

The set of feasible utility pairs for both initial types consistent with finite profits is[1]

$$\mathcal{V} \equiv \left\{\left(V_1\left(M \mid l\right), V_1\left(M \mid h\right)\right) \mid M \in \mathcal{M}_{IC}, \Pi(M) > -\infty\right\}.$$

I refer to the profit maximization problem of the firm, for any $V = (V_l, V_h) \in \mathcal{V}$ as

$$\Pi^*\left(V\right) \equiv \sup_{M \in \mathcal{M}_{IC}} \Pi\left(M\right), \tag{2}$$

subject to $V_1\left(M \mid \theta\right) = V_\theta$, for $\theta \in \{l, h\}$. If a unique solution exists, I denote it as M^V.

Complete information benchmark

In the absence of private information, efficient long-term contracts provide complete coverage with a consumption level that only depends on θ_1. The complete information problem is that of choosing $M \in \mathcal{M}$ to maximize $\Pi(M)$, subject to payoff constraints $V_1\left(M \mid \theta\right) = V_\theta$, for $\theta \in \{l, h\}$. The solution to the complete information problem is denoted by $\left\{z_t^{CI}\right\}_{t=1}^{T}$,

[1]Restricting attention to mechanisms with finite profits allows me to ignore transversality conditions in studying the firm's problem. It is without loss for the analyses in Sections 7 and 8 or if $T < \infty$.

and is described in Lemma 1. A full coverage flow contract with consumption level $c \in \mathbb{R}_+$ is denoted as z^c.

Lemma 1. *The solution to the complete information problem satisfies* $z_t^{CI}\left(\eta_p^{t-1}, \theta_t\right) = z^{c^{CI}(\theta_1)}$, *where* $c^{CI}(\cdot)$ *is defined by* $\sum_{t=1}^{T} \delta^{t-1} u\left(c^{CI}(\theta_1)\right) = V_{\theta_1}$.

If $V_l = V_h$, the complete information solution is incentive compatible and hence solves $\Pi^*(V)$. Our analysis will focus on utility pairs $V \in \mathcal{V}$ with $V_h > V_l$, i.e., where the utility delivered to an initially high-type consumer is higher than that of a low type. The focus on this case is justified in Sections 7 and 8, where this inequality is shown to hold in equilibrium for both the competitive and monopoly settings. In the reverse case where $V_l > V_h$, all the results presented hold when adequately interchanging the role of types h and l.

4 Incentives and distortions

I now characterize the solution to the profit maximization problem (2). The main result in this section, Lemma 3, provides a characterization of the solution of a relaxed problem, which is then shown to solve the firm's original profit maximization problem (Corollary 1).

In an optimal mechanism, the impact of any information revealed over time — be it signal realizations or type announcements — is determined by its likelihood ratio: the ratio of its probability when coming from either a low- or high-type. For example, income realizations that are relatively more likely for high-types are rewarded with higher flow and higher continuation utilities. Additionally, the assumption of type persistence implies that future high-type announcements, under truth-telling, are relatively more likely to come from a consumer with initially high-type. As a consequence, high-type announcements are rewarded with more attractive continuation contracts. More precisely, a solution to the relaxed problem is shown to satisfy three monotonicity properties that provide insights into the the efficient provision of dynamic incentives. These monotonicity properties are also used, in Subsection 4.4, to guarantee that the solution to the relaxed problem solves the original profit maximization problem.

4.1 Relaxed problem

A reporting strategy r is a one-shot critical deviation (OSCD) from truth-telling at period t with signal history ϕ^{t-1} if the consumer reports a high-type when having a low-type for the *first* time at period t with signals ϕ^{t-1}, but otherwise reports her type truthfully. In other words, it satisfies $r_t(\phi^{t-1}, h^{t-1}, h^{t-1}, l) = h$, but is otherwise identical to the truth-telling strategy. I say that a mechanism is one-shot incentive compatible (OSIC) if the consumer has no profitable OSCD, and the set of OSIC direct mechanisms is defined as $\mathcal{M}_{OSIC}$.

The relaxed problem is the following:

$$\Pi^R(V) \equiv \sup_{M \in \mathcal{M}_{OSIC}} \Pi(M), \tag{3}$$

subject to $V_1(M \mid \theta) = V_\theta$, for $\theta \in \{l, h\}$.

Assumption 1. *The relaxed problem has a solution, for all $V \in \mathcal{V}$.*

Existence of a solution is guaranteed in the presence of a finite horizon ($T < \infty$), or bounded utility function $u(\cdot)$. I do not provide an exhaustive analysis of the issue of existence, but my results apply more generally whenever an optimal mechanism exists.

Lemma 2. *If $T < \infty$ or $u(\cdot)$ is bounded, the relaxed problem has a solution for all $V \in \mathcal{V}$.*

Proof. Consider any $V \in \mathcal{V}$. The definition of $\mathcal{V}$ implies that $\Pi^R(V) \geq \Pi^*(V) > -\infty$. Consider a sequence of mechanisms $\left\{M^n = \{z_t^n\}_{t \leq T}\right\}_{n \in \mathbb{N}}$ feasible in problem $\Pi^R(V)$ such that $\Pi(M^n) > -\infty$ and $\Pi(M^n) \to^{n\to\infty} \Pi^R(V)$. The fact that $\Pi(M^n) > -\infty$ implies that, for each period t and history $(\phi^{t-1}, \theta^t) \in \Phi^{t-1} \times \theta^t$, sequence $\{z_t^n(\phi^{t-1}, \theta^t)\}_{n \in \mathbb{N}}$ is uniformly bounded by a constant $\bar{z}(\phi^{t-1}, \theta^t) < \sup_{c \geq 0} u(c)$ and, as a consequence, we can assume without loss (potentially using a subsequence) that $\{z_t^n(\phi^{t-1}, \theta^t)\}_{n \in \mathbb{N}}$ converges for any t, ϕ^{t-1}, θ^t. Now let $M^* \equiv \{\lim_n z_t^n\}_{t \leq T}$. If $u(\cdot)$ is bounded or $T < \infty$, the dominated convergence theorem implies that $V^r(M^*) = \lim_{n\to\infty} V^r(M^n)$ for any reporting strategy r, which implies that M^* is feasible in problem $\Pi^R(V)$. Fatou's lemma implies that (the profits of any two mechanisms only differ in the consumption paid out to the agent):

$$-\Pi(M^*) \leq \liminf_n \left[-\Pi(M^n)\right] = -\Pi^R(V), \tag{4}$$

which means that M^* solves $\Pi^R(V)$. □

Define $\tau(\eta^{t-1}, \theta_t)$ as the first period at which the consumer has a low type (if any), i.e.,

$$\tau(\eta^{t-1}, \theta_t) = \min\{t' \in \{0, \ldots, t\} \mid [\theta^{t-1}, \theta_t]_{t'} = l\},$$

with $\inf \emptyset = \emptyset$. For example, $\tau(\phi^t, (l, h, \ldots, h)) = 1$ represents a first period low-type, and $\tau = \emptyset$ represents the absence of a low-type realization, i.e., $\tau(\phi^t, (h, \ldots, h)) = \emptyset$. In the rest of this section, I omit the dependence of τ on the history of signals and types for brevity.

For any income level $y \in Y$ and signal $\phi_0 \in \Phi$, define likelihood ratios

$$\ell(y) \equiv \frac{p_l(y)}{p_h(y)} \text{ and } \ell(\phi_0) \equiv \frac{\sum_{y \in \phi^{-1}(\phi_0)} p_l(y)}{\sum_{y \in \phi^{-1}(\phi_0)} p_h(y)}.$$

These ratios are useful as they represent how informative each income/signal realization is in screening different types. A realization with high $\ell(\cdot)$ is more "indicative" of a low-type.

The following Lemma states that the relaxed problem has a unique solution and outlines key properties of its solution. This result is proved in the Appendix and its proof exploits the recursive structure of problem Π^R.

Lemma 3. *For any $V \in \mathcal{V}$, the relaxed problem has a unique solution. Moreover, if $V \in int(V)$ and $V_h > V_l$, its solution satisfies:*

(i) All OSIC constraints hold as equalities.

(ii) No distortions following low-type: there exists $\{c_t\}_1^T$, with $c_t : \Phi^{t-1} \mapsto \mathbb{R}_+$ such that $z_t(\phi^{t-1}, \theta^{t-1}, \theta_t) = z^{c_\tau(\phi^{\tau-1})}$, whenever $\tau \neq \emptyset$.

Additionally, there exist $\{\mu_t, \lambda_t\}_{t=1}^T$ with $(\mu_t, \lambda_t) : \Phi^t \mapsto \mathbb{R} \times \mathbb{R}_+$ such that, for any (η^{t-1}, θ_t) with $\tau = \emptyset$,

(iii) Flow contracts following high-types have partial coverage:

$$\mu_{t-1}(\phi^{t-1}) - \lambda_{t-1}(\phi^{t-1})\ell(y_t) \leq \frac{1}{u'\left(z_t(y_t \mid \eta^{t-1}, h)\right)} \tag{5}$$

with (5) holding as a equality if $z_t(y_t \mid \eta^{t-1}, \theta_t) > 0$.

(iv) Future type reward: $V_t(\eta^{t-1}, h) > V_t(\eta^{t-1}, l)$.

(v) Future signal effect: for any ϕ, ϕ' such that $\ell(\phi') \leq \ell(\phi)$:

$$\sum_{i=l,h} \pi_{li} V_t((\phi^{t-2}, \phi', h^{t-1}), i) \geq \sum_{i=l,h} \pi_{li} V_t((\phi^{t-2}, \phi, h^{t-1}), i).$$

Property (i) state that all upward incentive constraints considered in the relaxed problem hold as equalities. The optimal contract is supposed to provide higher utility to a consumer with high initial type, while deterring deviations from low-types. Given the presence of type persistence, a way to achieve this goal is to use a *future* high-type realization as a signal that the consumer's initial type is $\theta_1 = h$. In other words, the utility gap between consumers with initially high and low types is propagated to future periods, with consecutive high-type announcements being "rewarded" with higher utility. For this reason, upward incentive constraints bind not only at $t = 1$, but in all periods. Property (ii) corresponds to a "no distortion at the bottom" result. This follows directly from the fact that the relaxed profit maximization problem ignores "downward" incentive constraints. Properties (iii)-(v) of Lemma 3 illustrate how signal information and reports are used in an optimal mechanism to efficiently screen consumers with different types using both within-period coverage as well as continuation utilities. Subsections 4.2 and 4.3 provide an interpretation of these conditions, referred to as monotonicity conditions, and show their implications for incentive provision. In Subsection 4.4, they are used to show that the solution to the relaxed problem solves the firm's original profit maximization problem.

4.2 Flow monotonicity

Flow monotonicity is akin to standard static monotonicity notions. It guarantees that within-period incentives are aligned by making sure that the partial coverage contract tailored to the (current) high-type consumer rewards accident realizations that are indicative of a high type, based on its likelihood ratio. It corresponds to property (iii) in Lemma 3.

Definition. Flow contract z satisfies flow monotonicity if $\ell(y') > \ell(y) \implies z(y') \leq z(y)$.

The use of flow contracts satisfying flow monotonicity implies that consumers with high-types have a higher benefit from choosing them, relative to a full-coverage contract.

Lemma 4. *For any pair of flow contracts $z, z^c \in Z$, with z satisfying flow monotonicity and $c \geq 0$, $v(z,h) - v(z^c,h) \geq v(z,l) - v(z^c,l)$.*

Proof. First notice that $v(z^c,h) = v(z^c,l)$. Consider an ordering $\{y_{[k]}\}_{k=1}^{\#Y}$ of income realizations such that $\ell(y_{[k]})$ is weakly decreasing in k. Flow monotonicity implies that both $z(y_k)$ and $\left[1 - \ell(y_{[k]})\right]$ are increasing in k. Hence we have that

$$v(z,h) - v(z,l) = \sum_{k=1}^{K} p_h(y_k) u(z(y_k)) \left[1 - \ell(y_k)\right] \geq \left\{\sum_{k=1}^{K} p_h(y_k) u(z(y_k))\right\} \left\{\sum_{k=1}^{K} p_h(y_k) \left[1 - \ell(y_k)\right]\right\},$$

which is equal to zero since the last term in this expression is zero. □

4.3 Continuation monotonicity

Flow monotonicity is a static notion. In a dynamic environment firms can also use future contracts as additional screening instruments. From an incentive perspective, these dynamic incentive schemes are represented by changes to expected continuation utility within the mechanism as a response to either consumer choices or realized income within a period.

Continuation-signal-monotonicity (CSM) is a condition stating that signal realizations that are indicative of low-types are punished in future periods. This punishment is evaluated by comparing expected continuation utilities following different signal realizations, taking averages over possible future types using the probability distribution of a low-type, which should be discouraged from misreporting. It corresponds to property (iv) in Lemma 3.

Definition. For any $t < T$, a mechanism M satisfies CSM at period t and history η^{t-1} if, for any $y, y' \in Y$:
$\ell(\phi') \leq \ell(\phi)$ implies $\sum_{i=l,h} \pi_{li} V_t(M \mid (\phi^{t-2}, \phi', h^{t-1}), i) \geq \sum_{i=l,h} \pi_{li} V_t(M \mid (\phi^{t-2}, \phi, h^{t-1}), i)$.

In contrast, continuation-type-monotonicity (CTM), considers the impact of a type announcement on continuation utility, and requires that a mechanism deliver a higher continuation utility to a consumer with (current) high type. It corresponds to Lemma 3-(v).

Definition. For any $t < T$, a mechanism M satisfies continuation type monotonicity in period t and history η^{t-1} if $V_{t+1}(M \mid \eta^t, h) > V_{t+1}(M \mid \eta^t, l)$, for any $\eta^t = (\eta^{t-1}, \phi, h)$.

These monotonicity properties allows us to show that incentives are aligned: the continuation contract following an announcement $\hat{\theta} = h$ is most attractive to a consumer whose true type is $\theta_t = h$. For a fixed mechanism M, period $t < T$ and history $\eta^{t-1} = (\phi^{t-1}, \theta^{t-1})$, define the continuation utility obtained by a consumer with type $i \in \{l, h\}$ announcing $\hat{\theta}_t = h$ as

$$\hat{V}_{t+1}(i \mid \eta^{t-1}) \equiv \sum_{\phi \in \Phi} p_i(\phi) \left[\sum_{j=l,h} \pi_{ij} V_{t+1}\left((\phi^{t-1}, \phi, \theta^{t-1}, h), j\right) \right]. \tag{6}$$

Lemma 5. *If mechanism M satisfies CSM and CTM in period t, given history η^{t-1}, then $\hat{V}_{t+1}(h \mid \eta^{t-1}) \geq \hat{V}_{t+1}(l \mid \eta^{t-1})$.*

Proof. Rearranging summation terms gives us:

$$\begin{aligned} \hat{V}_{t+1}(h \mid \eta^{t-1}) - \hat{V}_{t+1}(l \mid \eta^{t-1}) &= \sum_{\phi \in \Phi} p_h(\phi)(\pi_{hh} - \pi_{lh}) \left[V_{t+1}(\eta^{t-1}, \phi, h, h) - V_{t+1}(\eta^{t-1}, \phi, h, l)\right] \\ &+ \sum_{\phi \in \Phi} p_h(\phi) \sum_{j=l,h} \pi_{lj} V_{t+1}(\eta^{t-1}, \phi, h, j) - \sum_{\phi \in \Phi} p_l(\phi) \sum_{j=l,h} \pi_{lj} V_{t+1}(\eta^{t-1}, \phi, h, j), \end{aligned}$$

Persistence of types and CTM implies that the first term is non-negative. The second and term terms can be rewritten as

$$\sum_{\phi \in \Phi} p_h(\phi)(1 - \ell(\phi)) \left[\sum_{j=l,h} \pi_{lj} V_{t+1}(\eta^{t-1}, \phi, h, j) \right],$$

which is positive since it is an average of the product of two positively correlated terms (from CSM), with the first term having zero expectation: $\sum_{\phi \in \Phi} p_h(\phi)(1 - \ell(\phi)) = 0$ □

4.4 Sufficiency of relaxed constraints

Besides providing insights into the efficient design of incentives, the three monotonicity notions introduced also allows us to guarantee that the solution of the relaxed problem is feasible in the original profit maximization problem, and hence solves it.

Lemma 6. *If M solves the relaxed problem, the consumer has no profitable one-shot deviation from truth-telling in M.*

Proof. Consider a period t with private history $\eta_p^{t-1} = (y^{t-1}, \hat{\theta}^{t-1}, \theta^{t-1})$ and current type θ_t. Since types follow a Markov process, the consumer's preferences over continuation reporting strategies are identical for (i) private history $\eta_p^{t-1} = (y^{t-1}, \hat{\theta}^{t-1}, \theta^{t-1})$ with period t type θ_t, and (ii) private history $\tilde{\eta_p}^{t-1} = (y^{t-1}, \theta^{t-1}, \theta^{t-1})$ with period t type θ_t. In other words, we only need to look for deviations for private histories without past misreports.

If θ^{t-1} includes any low-type realization, the result holds trivially since the optimal mechanism allocates constant consumption from period t onwards. I focus now on the case $\theta^{t-1} = h^{t-1}$.

If $\theta_t = l$, a one-shot deviation is a special case of the constraints considered in the OSCD concept and hence is satisfied in any mechanism that is feasible in the relaxed problem.

If $\theta_t = h$, using Lemma 3-(ii) we can represent the net gain from a one-shot deviation as

$$\frac{1-\delta^{T-t+1}}{1-\delta}u(c_t(\phi^{t-1})) - \left[v(z_t(\eta^{t-1}, h)) + \delta \hat{V}_{t+1}(h \mid \eta^{t-1})\right], \tag{7}$$

where $\eta^{t-1} = (\phi^{t-1}, \theta^{t-1})$ is the public history connected with the private history in focus and $\hat{V}_{t+1}$ is defined as in (6).

From Lemma 3, items (iii)-(v), we know that the optimal mechanism satisfies flow and continuation monotonicity and hence, using Lemmas 4 and 5, the net gain in (7) is weakly lower than

$$\frac{1-\delta^{T-t+1}}{1-\delta}u(c^t) - \left[v(z_t(\eta^{t-1}, l)) + \delta \sum_{\phi\in\Phi} p_l(\phi) \sum_{j=l,h} \pi_{lj}\hat{V}_{t+1}(l \mid \eta^{t-1})\right], \tag{8}$$

which is the net gain from truth-telling for a consumer with type $\theta_t = l$ relative to a misreport in period t. From Lemma 3-(i), we know this is zero. $\square$

The absence of profitable one-shot deviations is enough to guarantee that a mechanism $M = \{z_t\}_{t=1}^T$ is incentive compatible as long as the consumer's reporting problem satisfies continuity at infinity, which is guaranteed as long as flow utilities are bounded:

$$\sup\left\{|v(z_t(\eta^{t-1}), \theta)| \mid t = 1, \dots, T, \eta^{t-1} \in H^{t-1}, \theta \in \Theta\right\} < \infty,$$

in which case I say that the mechanism M is bounded. This condition is automatically guaranteed if time is finite or the Bernoulli utility function $u(\cdot)$ is bounded. It holds more generally if consumption in the solution to the relaxed problem is uniformly bounded.

Corollary 1. *If the solution to the relaxed problem is bounded, it is optimal.*

Proof. If a profitable deviation r from truth-telling r^* exists in mechanism M, boundedness of M implies — through standard arguments — that a finite profitable deviation r' only involving misreports up to period $t < \infty$ also exists. Since one-shot deviations from truth-telling are not profitable, then modifying r' by dictating truth-telling in period t, regardless of the history, is a weak improvement from r'. Repeated application of this argument implies that r^* weakly dominates r', a contradiction. Hence M solves the relaxed problem and is feasible in the firm's original problem, so it is optimal. □

5 Auxiliary problem

The characterization of coverage and price dynamics in this model poses two technical challenges. First, the flow contract space, $Z = \mathbb{R}_+^Y$, is multi-dimensional and does not have a natural notion of distortions, such as as under-provision in standard screening models (Mussa and Rosen (1978), Myerson (1981)). The underlying inefficiency in this model is the exposure of the consumer to risk, or partial coverage, which is introduced as a way of screening high-types. I introduce a notion of distortion that is directly tied to the source of distortions, which is the need to preclude low-type consumers from misreporting their types. Second, the introduction of risk aversion in a dynamic screening environment leads to a non-separability issue absent from linear environments. The marginal efficiency loss of from the introduction of distortions potentially depends on the underlying utility level obtained by the consumers. As a consequence, the optimal contract jointly chooses both the intertemporal allocation of utility to be provided to the consumer as well as the spreading of distortions over time. I tackle both of these issues by introducing a tractable auxiliary static cost minimization problem. This auxiliary cost function is then used to study the original dynamic profit maximization problem.

5.1 Definition

Flow contracts fulfill two roles in incentive provision: to provide an utility level to a truth-telling consumer and to discourage misreports. Hence I study an auxiliary cost minimization problem of finding an optimal flow contract $z \in Z$ satisfying a promise keeping constraint, as well as a threat-keeping constraint introducing a penalty for misreporting l-type consumers:

$$\chi(\nu, \Delta) \equiv \inf_{z \in Z} \sum_{y \in Y} p_h(y) \, z(y) \,,$$

subject to

$$v(z, h) = \nu \text{ and } v(z, l) = \nu - \Delta. \tag{9}$$

The set of pairs $(\nu, \Delta) \in \mathbb{R}^2$ such that a flow contract satisfying (9) exists is $A \subset \mathbb{R}^2$. The utility wedge Δ between different types is the source of distortions in the contract. I refer to it directly as the level of distortion in a cost-minimizing flow contract.

Lemma 7. *For any* $(\nu, \Delta) \in A$, $\mathcal{P}^A$ *has a unique solution and* χ *is strictly convex. If the solution at* (ν, Δ) *is interior then* χ *is twice differentiable in an open neighborhood of* (ν, Δ).

Proof. Follows directly from Lemma 15 in the Appendix. □

For any $(\nu, \Delta) \in A$, I denote the solution of $\mathcal{P}^A$ as $\zeta(\nu, \Delta)$. In the remainder of the analysis, I assume that this cost function is supermodular.

Assumption 2. *(Cost supermodularity) Function* $\chi(\cdot)$ *satisfies:* $\frac{\partial^2 \chi(\nu,\Delta)}{\partial\nu\partial\Delta} \geq 0$, *whenever* $\mathcal{P}^A$ *has an interior solution and* $\Delta \geq 0$.

The supermodularity of function $\chi(\cdot)$ can be written in terms of utility function $u(\cdot)$, and is equivalent to the requirement that the coefficient of absolute risk aversion $r_u(c) \equiv -\frac{u''(c)}{u'(c)}$ does not decrease with consumption "too quickly". This is guaranteed if $u(\cdot)$ has non-decreasing absolute risk aversion (IARA), which includes constant absolute risk aversion (CARA) utility as a special case. It also holds for the case of constant relative risk aversion (CRRA) utility with coefficient above 1/2. The following result is proved in the Appendix.

Lemma 8. *Assume* $r_u : \mathbb{R}_+ \to \mathbb{R}_+$ *is differentiable, then the following are equivalent:*
(i) Assumption 2 holds,
(ii) $r'_u(x)\, u'(x) + 2\,[r_u(x)]^2 \geq 0$,
(iii) $\psi'''(x) > 0$.

5.2 Interpretation

The marginal cost of utility and distortion in a cost-minimizing flow contract is connected to the risk exposure and level of consumption it provides to the consumer. This connection is useful as, in Section 6, I characterize the optimal intertemporal allocation of consumption and distortions in terms of marginal costs. Since $\chi(\cdot)$ is convex, an increase in flow utility ν (or Δ) leads to an increase in χ_ν (or in χ_Δ). Final consumption is related to marginal costs in an optimal interior flow contract $\zeta(\cdot)$ according to (see Lemma 16 for a proof):

$$\frac{1}{u'(\zeta(y))} = \chi_\nu(\nu, \Delta) + \chi_\Delta(\nu, \Delta)\,[1 - \ell(y)]\,, \tag{10}$$

where I use notation $\chi_s(\cdot) \equiv \frac{\partial \chi(\cdot)}{\partial s}$. From (10), we can see that the marginal cost of flow utility relates to the consumer's marginal utility through the following equation:

$$\sum_{y \in Y} p_h(y) \frac{1}{u'(\zeta(y))} = \chi_\nu(\nu, \Delta), \tag{11}$$

Since the function $[u'(c)]^{-1}$ is strictly increasing in consumption, an increase in χ_ν can be seen as an overall increase in consumption, with the exact connection being dependent on the shape of utility $u(\cdot)$. Equation (10) also implies the following relationship between the distortion level Δ and the consumer's risk exposure to income shocks:

$$\frac{1}{u'(\zeta(y'))} - \frac{1}{u'(\zeta(y))} = -\chi_\Delta(\nu, \Delta) \left[\ell(y') - \ell(y)\right].$$

Intuitively, cost minimizing flow contracts reward income realizations with low likelihood ratio $\ell(\cdot)$. These two marginal cost equations imply that

$$\chi_\Delta^2(\nu, \Delta) \sum_{\phi \in \Phi} p_h(y) \left[1 - \ell(y)\right]^2 = \sum_{\phi \in \Phi} p_h(y) \left\{ [u'(\zeta(y))]^{-1} - \sum_{y' \in Y} p_h(y')[u'(\zeta(y'))]^{-1} \right\}^2. \quad (12)$$

The right-hand side corresponds to the variance of the inverse marginal utility of the consumer. Hence an increase in distortion Δ, and hence in χ_Δ, generates a larger dispersion in the inverse marginal utility of consumer across income realizations. A larger distortion requires that the flow contract expose the consumer to more risk.

Example: CRRA preferences

If $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$, with coefficient of relative risk aversion $\rho > 0$, (11) and (12) imply that an increase in the flow utility ν leads to an increase in the ρ-th moment of consumption, while an increase in distortion Δ leads to a larger variance of the ρ-th power of consumption:

$$\chi_\nu(\nu, \Delta) = \sum_{y \in Y} p_h(y)\zeta(y)^\rho, \text{ and } \chi_\Delta^2(\nu, \Delta) = \frac{\sum_{y \in Y} \left[\zeta(y)^\rho - \mathbb{E}_h(\zeta^\rho)\right]^2}{\sum_{y \in Y} p_h(y) \left[1 - \ell(y)\right]^2},$$

where $\mathbb{E}_h(\zeta^\rho) \equiv \sum_{y \in Y} p_h(y)\zeta(y)$.

In the case $\rho = 1$, where the utility becomes $u(c) = \log(c)$, the marginal costs χ_ν and χ_Δ are multiples of the mean and variance, respectively, of the within-period consumption of the consumer. Alternatively, if $\rho = 1/2$, the marginal costs χ_ν and χ_Δ are multiples of the mean and variance, respectively, of the within-period ex-post utility of the consumer — $u(\zeta(y))$ — in the cost minimizing contract $\zeta(\cdot)$. Hence distortion parameter Δ represents the consumer's exposure to risk, but evaluated from the point of view of her utility level.

Example: binary outcomes

In Section 6, I provide results on the dynamics of utility flow and distortions. In the special case of binary outcomes, with $Y = \left\{\underline{y}, \bar{y}\right\}$ satisfying $\ell(\underline{y}) > \ell(\bar{y})$, condition (9) imply that both are connected to consumer's income-contingent utility in a simple way:

$$u(\zeta(\bar{y})) = \nu + \frac{p_h(\underline{y})}{p_h(\bar{y}) - p_l(\bar{y})}\Delta, \text{ and } u(\zeta(\underline{y})) = \nu - \frac{p_h(\bar{y})}{p_h(\bar{y}) - p_l(\bar{y})}\Delta.$$

6 Distortion and consumption dynamics

In this section, I use the auxiliary cost problem introduced in Section 5 to study the dynamics of distortions and consumption in the optimal mechanism. The firm's profit maximization is directly linked with problem $\mathcal{P}^A$ since each within-period flow contract provided by the firm must be optimal within the set of flow contracts that deliver the same utility level for truth-telling consumers (which corresponds to ν) and discourages misreporting consumers by the same amount (which corresponds to Δ). In other words, each contract solves the auxiliary problem $\mathcal{P}^A$ for a particular pair (ν, Δ). We can then separate the problem of the firm into two "sub-problems": one in which flow utility and distortions are allocated across periods, which I study in this section, and another one in which the chosen flow utility and distortion levels must be delivered in a cost-minimizing fashion, as discussed in Section 5.

The following result formalizes this separation, by showing that all flow contracts delivered in the optimal mechanism are solutions to the auxiliary cost minimization problem.

Given optimal mechanism $M = \{z_t\}_{t=1}^T$, define the following:

$$\nu_t(\phi^{t-1}) \equiv v\left(z_t(h^{t-1}, \phi^{t-1}, h), h\right) \text{ and } \Delta_t(\phi^{t-1}) \equiv \nu_t(\phi^{t-1}) - v\left(z_t(h^{t-1}, \phi^{t-1}, h), l\right),$$

which correspond respectively to the flow utility and distortion in the partial coverage contract offered in period t following type announcements h^t. I use a similar notation to refer to the flow per-period utility derived by the consumer following a first low-type announcement in period t, i.e., $\nu_t^l(\phi^{t-1}) \equiv u(c_t(\phi^{t-1}))$.

Proposition 1. *The optimal mechanism $M = \{z_t\}_{t=1}^T$ satisfies, for any $t = 1, \ldots, T$,*

$$z_t(h^{t-1}, \phi^{t-1}, h) = \zeta(\nu_t(\phi^{t-1}), \Delta_t(\phi^{t-1})).$$

Proof. Fix any period $t < T$ and signal history ϕ^t. For any mechanism $\tilde{M} = \{\tilde{z}_t\}_{t=1}^T$, the flow contract $\tilde{z}_t(h^{t-1}, \phi^{t-1}, h)$ only affects the OSIC constraints in the relaxed problem via $v\left(\tilde{z}_t(h^{t-1}, \phi^{t-1}, h), h\right)$ and $v\left(\tilde{z}_t(h^{t-1}, \phi^{t-1}, h), l\right)$, while it only affects the firm's profits via

$$\sum_{y \in Y} p_h(y)\, \tilde{z}_t(h^{t-1}, \phi^{t-1}, h)(y).$$

Hence modifying mechanism M by substituting flow contract $z_t(h^{t-1}, \phi^{t-1}, h)$ by $\zeta(\nu_t(\phi^t), \Delta_t(\phi^t))$

still satisfies OSIC and (given uniqueness of solution in $\mathcal{P}^A$) strictly increases profits in the relaxed problem if $z_t(h^{t-1}, \phi^{t-1}, h) \neq \zeta(\nu_t(\phi^t), \Delta_t(\phi^t))$. □

From now on, I refer to ν_t and Δ_t simply as the flow utility and distortion in the optimal mechanism following a sequence of high-type announcements. We can now use Proposition 1 and focus on the problem of intertemporal allocation of flow utility ν and distortion Δ in the optimal contract. The following proposition, proved in the Appendix, displays the optimality conditions connected with intertemporal allocation of utility and distortions. Since all terms in Proposition 2 depend on ϕ^{t-1}, I omit this dependence for brevity.

Proposition 2. *For any $t < T$ and $\phi^{t-1} \in \Phi^{t-1}$, if the solution to the cost minimization problem $\mathcal{P}^A$ has an interior solution, the following hold*

$$\chi_\nu(\nu_t, \Delta_t) = \sum_{\phi \in \Phi} p_h(\phi) \left\{ \pi_{hh} \chi_\nu[(\nu_{t+1}, \Delta_{t+1})(\phi)] + \pi_{hl} \chi_\nu[(\nu^l_{t+1}(\phi), 0)] \right\}, \tag{13}$$

$$\begin{aligned} \chi_\Delta(\nu_t, \Delta_t) = \sum_{\phi \in \Phi} p_h(\phi) \frac{\pi_{hh}}{\pi_{hh} - \pi_{lh}} \Big\{ & \chi_\Delta[(\nu_{t+1}, \Delta_{t+1})(\phi)] \\ & + \pi_{hl} \left[\chi_\nu[(\nu_{t+1}, \Delta_{t+1})(\phi)] - \chi_\nu[(\nu^l_{t+1}(\phi), 0)] \right] \Big\}. \end{aligned} \tag{14}$$

Proposition 2 relies on the use of local optimality conditions of the firm's profit maximization problem. The interiority condition in Proposition 2 holds if the optimal mechanism has strictly positive consumption. Conditions (13) and (14) characterize the dynamics of flow utility and distortions along histories involving distortions.

Equation (13) represents the efficient intertemporal allocation of flow utilities, or consumption. It states that the marginal cost of flow utility for a high-type in period t must be equalized to the expected marginal cost of flow utility in period $t + 1$. The marginal cost of flow utility coincides with the expectation of the inverse of the consumer's marginal utility (see Subsection 5.2), which means that this condition is a special case of the celebrated inverse Euler equation (see Rogerson (1985), Farhi and Werning (2012)).

Equation (14), on the other hand, represents the efficient intertemporal allocation of distortions. Let's consider the problem, in period $t < T$, of discouraging a consumer with low period t type from pretending to have a high type. Due to the presence of type persistence, this can be done in two ways: by exposing the consumer to risk in period t, which is represented by a larger Δ_t; or by exposing the consumer to more risk in period $t + 1$ as long as the consumer still claims to be of high type ($\theta_{t+1} = h$). The optimal mechanism uses both screening methods in a balanced way. Equation (14) illustrates the differences in using current versus future period distortions. First, notice that future distortions are only useful due to the persistence of types. As type persistence is reduced, or $(\pi_{hh} - \pi_{lh})$

is smaller, the optimal mechanism relies mostly on current period distortions for screening purposes. In the limit where $\pi_{hh} = \pi_{lh}$, the use of future distortions in screening is useless and, as a consequence, the optimal contract only features distortions in the first period. Second, reducing the distortion in period t while increasing it in period $t+1$ not only has a direct cost impact as it requires exposing the consumer to risk (this is captured by the χ_Δ term), but it also implies that next period's type realization now has a larger impact on the consumer's final utility. This additional utility dispersion has a marginal cost which depends on the gap of the marginal cost of flow utility in period $t+1$ for both possible types (represented by the second and third terms on the right-hand side of (14)).

6.1 Realization-independent contracts

I start by focusing on the case of realization-independent contracts, i.e., long-term contracts that do not use the history of past income realizations when determining the menu to be offered to the consumer within a period. As discussed in the introduction, this represents an extreme case of price restrictions that limit how much information can be used explicitly by firms when pricing consumption. This subsection also restricts attention to the finite-horizon case, i.e., $T < \infty$.

Since flow contracts now only depend on the history of announcements, I drop the dependence on history ϕ^t. In this case, equations (13) and (14) become

$$\chi_\nu(\nu_t, \Delta_t) = \pi_{hh}\chi_\nu(\nu_{t+1}, \Delta_{t+1}) + \pi_{hl}\chi_\nu(\nu^l_{t+1}, 0), \text{ and} \tag{15}$$

$$\chi_\Delta(\nu_t, \Delta_t) = \frac{\pi_{hh}}{\pi_{hh} - \pi_{lh}} \left\{ \chi_\Delta(\nu_{t+1}, \Delta_{t+1}) + \pi_{hl} \left[\chi_\nu(\nu_{t+1}, \Delta_{t+1}) - \chi_\nu(\nu^l_{t+1}, 0) \right] \right\}. \tag{16}$$

The supermodularity guaranteed by Assumption 2 allows us to characterize the dynamic behavior of utility and distortions. Let's focus on the intertemporal allocation between a given periods t and $t+1$, following a sequence of high-type realizations h^t. Given that the optimal contract rewards high-type announcements, the flow utility following an additional high-type realization in period $t+1$, ν_{t+1}, is higher relative to that of a consumer with a low-type realization, ν^l_{t+1}.

Together with the presence of distortions following a high-type announcement ($\Delta_{t+1} > 0$) and supermodularity of $\chi(\cdot)$ (Assumption 2), we can then conclude that $\chi_\nu(\nu_{t+1}, \Delta_{t+1}) > \chi_\nu(\nu^l_{t+1}, 0)$. From (15), we can conclude that consecutive high-type announcements lead to an increase in the marginal cost of flow utility: $\chi_\nu(\nu_{t+1}, \Delta_{t+1}) > \chi_\nu(\nu_t, \Delta_t) > \chi_\nu(\nu^l_{t+1}, 0)$. Now, using intertemporal distortion allocation condition (16), we have that

$$\chi_\Delta(\nu_t, \Delta_t) > \frac{\pi_{hh}}{\pi_{hh} - \pi_{lh}} \chi_\Delta(\nu_{t+1}, \Delta_{t+1}) > \chi_\Delta(\nu_{t+1}, \Delta_{t+1})$$

In other words, profit maximization mandates that, along high-type path h^T, the marginal cost of flow utility is increasing while the marginal cost of distortions is decreasing. Using convexity of χ and, once again, Assumption 2 (see Lemma 17) these properties can be translated into statements about the dynamic behavior of flow utility and distortions, which are summarized in the following Proposition, proved in the Appendix.

Proposition 3. *If $V_h > V_l$, the optimal mechanism satisfies:*
(i) High-type utility flows increase, i.e., $\{\nu_t\}_{t=1}^T$ is strictly increasing,
(ii) Distortions decrease, i.e., $\{\Delta_t\}_{t=1}^T$ is strictly decreasing.

6.2 Realization-dependent contracts

For general signal structures, I extend the monotonicity result of Proposition 3 in two cases.

6.2.1 Quadratic cost

First, consider the case of constant absolute risk aversion with coefficient 1/2 as an illustrative example, i.e., $u(c) = 2\sqrt{c}$. This knife-edge example simplifies the analysis since it is the only one in which the marginal costs of flow utility and distortions are separable, i.e., $\frac{\partial^2 \chi(\nu,\Delta)}{\partial \nu \partial \Delta} = 0$. Moreover, marginal costs $\chi_\nu(\cdot)$ and $\chi_\Delta(\cdot)$ are linear. In this case, intertemporal optimality conditions (13) and (14) in Proposition (2) can be rewritten as statements in terms of utility flow and distortions: for constant $N \equiv \frac{\partial^2 \chi}{\partial \nu^2} > 0$,

$$\nu_t(\phi^{t-1}) = \sum_{\phi \in \Phi} p_h(\phi_t) \left[\pi_{hh}\nu_{t+1}(\phi^t) + \pi_{hl}\nu_{t+1}^l(\phi^t)\right], \tag{17}$$

$$\Delta_t(\phi^{t-1}) = \sum_{\phi \in \Phi} p_h(\phi_t) \frac{\pi_{hh}}{\pi_{hh} - \pi_{lh}} \Delta_{t+1}(\phi^t) + N\left[\nu_{t+1}(\phi^t) - \nu_{t+1}^l(\phi^t)\right]. \tag{18}$$

The first equation implies that, flow utilities form a martingale in the optimal mechanism. In the optimal mechanism, the continuation utility following a high-type realization is always larger than the one following a low-type announcement, and the martingale property of flow utilities implies that *flow utilities* in any given period t are also larger for a high-type, relative to a low-type. It follows from (17) that flow utility dynamics are similar to the realization-independent case: a high-type (low-type) realization leads to higher (lower) utility flow – averaging over the possible signal realizations. Together (18), this implies that distortions follow a supermartingale, conditional on the type path θ^T.

Proposition 4. *If $V_h > V_l$ and the optimal mechanism is interior, then:*

(i) distortions follow a supermartingale, conditional on θ^T, i.e.,

$$\Delta_t(\phi^{t-1}) > \sum_{\phi\in\Phi} p_h(\phi_t)\Delta_{t+1}(\phi^t)$$

and (ii) within period flow utilities are increasing in past periods' type announcements, i.e.,

$$\sum_{\phi\in\Phi} p_h(\phi_t)\nu_{t+1}(\phi^t) > \nu_t(\phi^{t-1}) > \sum_{\phi\in\Phi} p_h(\phi_t)\nu^l_{t+1}(\phi^t)$$

Proof. First notice that the flow utility in period t if the consumer's first low-type announcement is in period t is strictly lower than that if the consumer has one more high-type realization in period t since $V_t(\phi^{t-1}, h^{t-1}, l)$ (generated by constant consumption) is equal to

$$\begin{aligned}
\nu^l_t(\phi^{t-1})\sum_{\tau=t}^{T}\delta^{\tau-t} =& \nu_t(\phi^{t-1}) + \delta\sum_{\phi\in\Phi} p_l(\phi_t)\left[\pi_{lh}V_{t+1}(\phi^t, h^t, h) + \pi_{ll}V_{t+1}(\phi^t, h^t, l)\right] - \Delta_t(\phi^{t-1})\\
<& \nu_t(\phi^{t-1}) + \delta\sum_{\phi\in\Phi} p_h(\phi_t)\left[\pi_{hh}V_{t+1}(\phi^t, h^t, h) + \pi_{hl}V_{t+1}(\phi^t, h^t, l)\right] - \Delta_t(\phi^{t-1})\\
=& \nu_t(\phi^{t-1})\sum_{\tau=t}^{T}\delta^{\tau-t} - \Delta_t(\phi^{t-1}).
\end{aligned}$$

The first equality corresponds to the binding incentive constraint in period t. The inequality follows from Lemma 3-(iv). Finally, the last equality follows from (17). Hence we conclude that $\nu_t(\phi^{t-1}) > \nu^l_t(\phi^{t-1})$, for all t and $\phi^{t-1} \in \Phi^{t-1}$. The result then follows directly from equations (17) and (18). □

6.2.2 General preferences

Now consider an arbitrary signal structure with $T = 2$. In maximizing profits the firm has an incentive to reward subsequent high-type announcements, and to rely less on later periods' distortions in order to provide incentives. As a consequence, the marginal costs of flow utility must be increasing, while the marginal cost of distortions must be decreasing following consecutive high-type announcements (which correspond to partial coverage), as shown below. The interpretation of Proposition 5, using the results from Subsection 5.2, is that the consecutive choices of partial coverage are rewarded, when averaging over signal realizations, with more consumption and less distortions.

Proposition 5. *Assume $T = 2$, the optimal mechanism satisfies*

$$\chi_\Delta(\nu_1, \Delta_1) > \sum_{\phi \in \Phi} p_h(\phi)\chi_\Delta[(\nu_2, \Delta_2)(\phi)],$$

$$\sum_{\phi \in \Phi} p_h(\phi)\chi_\nu[(\nu_2, \Delta_2)(\phi)] > \chi_\nu(\nu_1, \Delta_1) > \sum_{\phi \in \Phi} p_h(\phi)\chi_\nu[(\nu_2^l(\phi), 0)].$$

Proof. Lemma 3 implies that $V_2(\phi, h) = \nu_2(\phi, h) > V_2(\phi, l) = \nu_2^l(\phi)$, which implies that $\chi_\nu[(\nu_2, \Delta_2)(\phi)] > \chi_\nu[\nu_2^l(\phi)]$, for all signals $\phi \in \Phi$. The result follows from (13) and (14). □

7 Competitive analysis

I now consider a competitive model that extends the analysis of Rothschild and Stiglitz (1976) and Cooper and Hayes (1987) in allowing for both persistent non-constant risk types and offers containing dynamic mechanisms. For now, I assume a single contracting stage between consumers and firms in the first period, which means that both firms and consumers can commit. The role of the commitment assumption is discussed in Subsection 7.1.

Consider the following extensive form. A finite set of firms simultaneously offer a mechanism to a consumer. The consumer observes her initial type θ_1 and decides which firm's mechanism to accept, if any. I assume exclusivity, i.e., the consumer can choose at most one mechanism. If the buyer does not accept any offer, she obtains no insurance coverage and gets discounted utility

$$\underline{V}_i \equiv \mathbb{E}\left[\sum_{t=1}^{T} \delta^{t-1} u\left(y_t\right) \mid \theta_1 = i\right]. \tag{19}$$

If the consumer accepts a contract, in each period she observes type $\theta_t \in \Theta$, then announces a message to the chosen firm. At the end of the period the income realization y_t is observed and the customer receives (or pays) transfers from the firm as described in the chosen mechanism. I study (weak) Perfect Bayesian Bayesian (PBE) of this extensive form.

If the consumer's types were observed by firms, the consumer would obtain actuarily fair full insurance in equilibrium, smoothing consumption both across income realizations within a period as well as across periods. In other words, a consumer with initial type $\theta \in \{l, h\}$ would receive a time- and income-independent flow consumption with total discounted utility

$$V_i^{FI} \equiv u(c_i^{FI}) \sum_{t=1}^{T} \delta^{t-1}, \tag{20}$$

where c_i^{FI} represents the discounted average expected lifetime income of a consumer with initial type $\theta_1 = i$:

$$c_i^{FI} \equiv \mathbb{E}\left[\frac{\sum_{t=1}^{T}\delta^{t-1}y_t}{\sum_{t=1}^{T}\delta^{t-1}} \mid \theta_1 = i\right].$$

Equilibrium outcomes are reminiscent of RS, with (initially) low-type consumers receiving their full-information utility level efficiently, while (initially) high-type consumers receive an inefficient partial coverage contract which delivers utility in the interval $\left(V_l^{FI}, V_h^{FI}\right)$. I assume that the full information continuation utility vector is feasible in the presence of private information, i.e., that $(V_l^{FI}, V_h^{FI}) \in \mathcal{V}$. This assumption insures that the following critical payoff vector, which will be shown to correspond to the equilibrium utility level of the consumer, is well-defined. Define as $\Pi_i^*(V)$, for $i = l, h$, the maximal expected discounted profit obtained by the firm conditional on the consumer's initial type being $\theta_1 = i$.

Lemma 9. *There exists a unique pair* $V^* \equiv (V_l^*, V_h^*) \in \mathcal{V}$ *satisfying:* $V_l^* = V_l^{FI}$ *and* $\Pi_h\left(V_l^*, V_h^*\right) = 0$.

Proof. Utility level V_l^* can be defined as V_l^{FI}. The existence and uniqueness of V_h^* follows from the fact that $\Pi_h\left(V_l^*, \cdot\right)$ is strictly decreasing and continuous (since it is convex) and satisfies $\Pi_h\left(V_l^*, V_l^*\right) > 0$ and $\Pi_h\left(V_l^*, V_l^{FI}\right) < 0$. □

For simplicity, I also focus on equilibria with two properties.[2] First, the consumer's strategy is symmetric, i.e., the probability she accepts the offer of firm F only depends on firm F's offer and the set of offers, but not on j's identity. Second, I assume that the consumer follows the truth-telling reporting strategy whenever it is optimal.

I say that a mechanism $M = \{z_t\}_{t=1}^{T}$ constitutes an equilibrium outcome if a PBE exists in which the on-path net consumption of the consumer corresponds exactly to their consumption in mechanism M when following a truth-telling reporting strategy. The following result — proven in the Appendix — shows that the unique equilibrium outcome is characterized by the solution of problem $\Pi(\cdot)$ studied in Sections 4-6.[3]

Proposition 6. *Any pure strategy PBE has outcome* M^{V^*}*. Moreover, a pure strategy equilibrium exists if, and only if,*

$$u'\left[u^{-1}\left(c_l^{FI}\right)\right]\frac{\partial_+\Pi_h\left(V^*\right)}{\partial V_l} < \frac{\pi_l}{\pi_h}. \tag{21}$$

The left-hand side of condition (21) does not depend on initial type distribution (π_l, π_h), and hence a pure strategy equilibrium exists if, and only if, the share of (initially) low-types in the population is sufficiently large. This is in line with the classical analysis in RS,

[2] These two restrictions do not affect equilibrium outcomes but substantially simplify the analysis.

[3] We use notation $\frac{\partial_+\Pi}{\partial V_i}(V)$ to denote the right-derivative of Π with respect to V_i at V.

which requires that the share of high-risk consumers be sufficiently high. Since $V_l^* < V_h^*$, the equilibrium outcome in the competitive model — described in Proposition 6 — is the solution of a particular instance of the problem studied in Sections 4-6.[4]

7.1 The role of commitment

In practice, consumer commitment may be the result of frictions that hinder consumers' contract switching, such as switching costs (see Honka (2014) and Handel and Schwartzstein (2018)). An alternative justification for commitment is informational: consumers' decision to search for a new contract may be a negative signal and lead to less attractive offers for switching consumers. I now show that, if consumer's **low-type is an absorbing state** ($\pi_{\mathbf{ll}} = \mathbf{1}$), firm's negative inference from consumer's switching decision serves as a commitment device.

Consider the following extension of the model discussed in Section 7. A different finite set of firms $\mathcal{F}_t$ can make offers to the consumer in each period. In the first period, the consumer and firms $F \in \mathcal{F}_1$ interact as described in the baseline competitive model. However, in each period $t = 2, \ldots, T$ the consumer decides whether to stay with her current mechanism or reenter the market searching for a new contract. If the consumer decides to reenter the market, firms $F \in \mathcal{F}_t$ observe the consumer reentry decision and then simultaneously offer mechanisms to the consumer. The consumer then decides whether to accept any of the new offers made or to remain uninsured. I restrict attention to realization-independent contracts.

The outcome described in Proposition 6 can arise in a PBE of this extended model, as long as firms' off-path beliefs are pessimistic. Firms' offers upon reentry in periods $t = 2, \ldots, T$ depend on their beliefs about the type of a reentering consumer. The most pessimistic belief firms may hold is to believe the consumer has a low type in the current period for sure. In this case, it is optimal for firms $F \in \mathcal{F}_t$ to behave as if they were in a market without information asymmetry and offer an efficient contract which provides perfect consumption smoothing for a consumer with low-type in period t, i.e., with consumption flow $c^o \equiv \mathbb{E}\left[\tilde{y}_t \mid \tilde{\theta}_t = l\right]$. These strategies are sequentially rational for firms given their beliefs. Consider the strategy profile for firms such that firms $F \in \mathcal{F}_1$ use the strategy described in the commitment model, while firms $F \in \mathcal{F}_t$ for $t \geq 2$ make the constant consumption offer with consumption c^o. Define $V_t^o \equiv u(c^o) \sum_{\tau=t}^{T} \delta^{\tau-t}$ to be the continuation utility from taking such an offer. The following lemma shows that, given firms' strategy profile, consumers never want to reenter the market.

Lemma 10. *If the equilibrium contract M^{V^*} is interior, then the consumer's continuation utility satisfies $V_t(\theta^t) \geq V_t^o$ for all periods $t = 1, \ldots, T$, and all $\theta^t \in \Theta^t$.*

[4] The interiority of the equilibrium outcome, which is assumed in multiple characterization results, can be guaranteed as long as the income distribution induced by both types is not "too" distinct. To see this notice that, if $p_h = p_l$, we have that $V_l^* = V_l^*$ and as a consequence the equilibrium outcome is the full information one, which is interior.

Proof. First notice that, from Proposition 6 and $\pi_{ll} = 1$ we have that $V_1(l) = \sum_{\tau=1}^{T} \delta^{\tau-1} u(c^o)$. Now consider period $t = 2, \ldots, T$ and history (h^{t-1}, l). The consumer's continuation utility in this case is given by $V_t(h^{t-1}, l) = \sum_{\tau=t}^{T} \delta^{\tau-t} (\nu_\tau - \Delta_\tau)$.

Since, from Proposition 3, $\{\nu_t\}_{t=1}^{T}$ is increasing and $\{\nu_t\}_{t=1}^{T}$ is decreasing, it follows that

$$V_t(h^{t-1}, l) > V_1(l) \frac{\sum_{\tau=t+1}^{T} \delta^{\tau-t-1}}{\sum_{\tau=1}^{T} \delta^{\tau-1}},$$

and hence the left-hand side is larger than $V_1(l) = V_t^o$. The only possible histories remaining are h^t, for $t = 1, \ldots, T$ and the proof is concluded since $V_t(h^t) > V_t(h^{t-1}, l)$. □

8 Monopoly

I now consider the Monopolist's problem of designing a mechanism to maximize revenue, with the assumption that the consumer is privately informed about their initial type at the contracting stage. All future type realizations are privately observed by the consumer. I assume $\underline{V}_h > \underline{V}_l$. For any utility level delivered to the consumer, conditional on her initial type, the offered mechanism must solve $\Pi(V)$, with $V \in \mathcal{V}$ solving problem $\mathcal{P}^M$:

$$\max_{V \in \mathcal{V}} \Pi(V), \text{ subject to } V_i \geq \underline{V}_i, \text{ for } i = l, h.$$

The next proposition shows that the participation constraint for the initially high-type buyer binds while the low-type buyer's participation constraint may be slack, i.e., information rents may be positive. This can be optimal as increasing the low-type's utility relaxes the incentive constraints in the profit maximization problem: $\Pi_h(\cdot)$ increases with V_l. We define $\underline{c}_i$ to be the constant consumption flow that generates discounted utility $\underline{V}_i$.

Proposition 7. *The Monopolist's optimal offer is M^{V^M}, where V^M is the solution to $\mathcal{P}^M$. Moreover, V^M satisfies $V_h^M = \underline{V}_h$ and $V_l^M \in [\underline{V}_l, \underline{V}_h)$.*

Proof. The first part of the proposition is trivial. We now prove that $V_h^M = \underline{V}_h$. By way of contradiction, suppose that $V_h^M > \underline{V}_h$. If, additionally, $V_l^M \geq V_h^M$, then mechanism $M^{V'}$, with $V' = V^M(1-\gamma) + \gamma u(0) \sum_{t=1}^{T} \delta^{t-1}$ for $\gamma > 0$ sufficiently small is feasible and a strict improvement given concavity of $\mathcal{V}$ and $\Pi(\cdot)$. Alternatively, if $V_l^M < V_h^M$, mechanism $M^{V'}$, with $V' = (V_l^M, V_h^M - \varepsilon)$, for $\varepsilon > 0$ sufficiently small, is feasible since it is in the convex hull of $\left\{V^M, (V_l^M, V_l^M)\right\} \subset \mathcal{V}$. It also strictly increases profits, contradicting optimality of V^M.

We now show that $V_l^M < \underline{V}_h$. If, by way of contradiction, $V^M = (\underline{V}_h, \underline{V}_h)$, reducing the low-type's utility is feasible and profitable as $\frac{\partial \Pi}{\partial V_l}(\underline{V}_h, \underline{V}_h) = -\pi_l \psi'(\underline{c}_h)$ (see Lemma 13). □

Proposition 7 allows us to obtain a simple characterization of information rents in the optimal contract. Given the concavity of $\Pi(\cdot)$, the utility vector $\underline{V} \equiv (\underline{V}_l, \underline{V}_h)$ solves problem $\mathcal{P}^M$ if, and only if,

$$\frac{\partial_+ \Pi}{\partial V_l}(\underline{V}) = -\frac{\pi_l}{u' \left[u^{-1}\left(\underline{c}_l\right)\right]} + \pi_h \frac{\partial_+ \Pi_h}{\partial V_l}(\underline{V}) \leq 0. \tag{22}$$

The first term in (22) corresponds to $\frac{\partial \Pi_l}{\partial V_l}(\underline{V})$, which follows from the fact that the low-type receives an efficient continuation contract. This result is summarized in the following result.

Proposition 8. *The Monopolist's optimal mechanism leaves no information rents (i.e., $V^M = \underline{V}$) if, and only if, (22) holds.*

In summary, the high-type receives no information rent and a distorted allocation as the least willing to pay for coverage. On the other hand, the low-type, who is most willing to pay for coverage, receives an efficient continuation contract. Additionally, if an initial low-type is sufficiently likely, this type receives no information rent in the optimal mechanism. The optimal mechanism *always* separates different initial types into different continuation contracts, even if both participation constraints bind.

9 Conclusion

This paper studies a natural dynamic extension of the workhorse model of adverse selection in insurance markets. The analysis introduces new tools to the dynamic mechanism design literature to deal with the presence of curvature and exploit the recursive structure of the profit maximization problem, and presents novel insights into the relationship between incentives and consumption/coverage dynamics.

The time-invariance assumptions used can be relaxed. All results can be extended to to time-dependent transitions $\left\{\pi_{ij}^t\right\}_{i,j,t}$, as long as types are persistent, i.e., $\pi_{ii}^t > \pi_{ji}^t$ for $t = 1, \ldots, T$. Allowing for time-dependent income distributions $\left\{p_i^t(\cdot)\right\}_{i,t}$ only affects the results in Section 6, which use the time-invariance of the auxiliary cost problem $\mathcal{P}^A$.

Section 7 focuses on pure strategy equilibria, which may not exist. In this case, equilibria will involve randomization over mechanisms, as in Farinha Luz (2017). Even if firms strategies imply the discounted utility generated by their mechanisms are random, each possible offer must be optimal given the utility level generated by it, i.e., it must be a solution to Π^*.

The recursive characterization of monotonicity, in terms of flow allocation and continuation utilities, may prove useful in the study of other dynamic mechanism design problems with curvature and persistence, such as screening risk averse buyers or Mirrleesian taxation.

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Appendix

1 Recursive formulation

In this section, I provide a recursive representation of the firm's relaxed problem described in Subsection 4.1. This representation is then used to prove Lemma 3. To explore the recursive structure of the problem at hand, I define, for any period, the set of feasible continuation utilities that can be delivered to a consumer for any given current type as well as the maximal continuation profit that can be obtained by the firm.

1.1 Definitions

For tractability, I will represent direct mechanisms through the utility flow generated for each period and history. Consider any direct mechanism $M = \{z_t\}_{t=1}^{T}$, period t and sequence of types and incomes $(\theta^t, y^t) \in \Theta^t \times Y^t$. The utility flow generated in period t is $\vartheta_t(\theta^t, y^t) = u(z_t(y_t \mid \eta^{t-1}, \theta_t))$, where $\eta^{t-1} = (\theta^{t-1}, (\phi(y_1), ..., \phi(y_{t-1})))$. I define a utility-direct mechanism (UDM) as a sequence of functions $M_1^u \equiv \{\vartheta_t\}_{t=1}^{T}$, with $\vartheta_t : \Theta^t \times Y^t \mapsto u(\mathbb{R})$, such that $\vartheta_t(\theta^t, y^t)$ is $(\eta^{t-1}, \theta_t, y_t)$-measurable. In other words, I impose that the utility flow in period t only depend on the history of incomes y^{t-1} through the observable signal history ϕ^{t-1}. Similarly, a period t continuation UDM assigns utility flows beyond period t which depend on the history of types and income levels starting at period t, i.e., it is equal to $M_t^u \equiv \{\vartheta_\tau\}_{\tau=t}^{T}$ such that $\vartheta_\tau : \Theta^{\tau-t} \times Y^{\tau-t} \mapsto u(\mathbb{R}_+)$ and $\vartheta_\tau([\theta^\tau]_t^\tau, [y^\tau]_t^\tau)$ is $\left([\eta^\tau]_t^{\tau-1}, \theta_\tau, y_\tau\right)$-measurable. I define the set of period t continuation UDM such that no one-shot OSCD with a misreport in periods $\tau \geq t$ is profitable as $\mathcal{M}_{t,OSIC}^u$.

Define, for any $t = 1, ..., T$, the set of type-contingent continuation utilities generated by incentive compatible continuation mechanisms:

$$\mathcal{V}_t \equiv \left\{ (V_l, V_h) \in \mathbb{R}^2 \mid \begin{array}{c} \exists M_t^u \equiv \{\vartheta_\tau\}_{\tau=t}^{T} \in \mathcal{M}_{t,OSIC}^u \text{ s.t.} \\ V_i = \mathbb{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \vartheta_\tau([\theta^\tau]_t^\tau, [y^\tau]_t^\tau) \mid \theta_t = i\right], \text{ for } i = l, h. \end{array} \right\},$$

with $\mathcal{V}_{T+1} \equiv \{(0,0)\}$ if $T < \infty$. It is easy to show that these sets are convex. For any $V \in \mathcal{V}_t$, define the problem $\mathcal{P}_t(V)$ of finding the maximal continuation profit obtained by a firm, conditional on continuation utility levels V, as (if $t = 1$, the expectation is unconditional)

$$\Pi_t(V) \equiv \sup_{M_t^u \in \mathcal{M}_{t,OSIC}^u} \mathbb{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \left[y_t - \psi\left[\vartheta_\tau([\theta^\tau]_t^\tau, [y^\tau]_t^\tau)\right]\right] \mid \theta^{t-1} = h^{t-1}\right], \tag{23}$$

subject to, for $i \in \{l, h\}$,

$$V_i = \mathbb{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \vartheta_\tau([\theta^\tau]_t^\tau, [y^\tau]_t^\tau) \mid \theta^t = \left(h^{t-1}, i\right)\right], \tag{24}$$

with $\Pi_{T+1}(0,0) = 0$ if $T < \infty$. Let $M_t^{*,V}$ be the solution to this problem, whenever well-defined. For $v \in \frac{1-\delta^{T-t+1}}{1-\delta} u(\mathbb{R}_+)$, define the full information continuation profit as

$$\Pi_{t,i}^{FI}(v) \equiv \sum_{\tau=t}^{T} \delta^{\tau-t} \left[\mathbb{E}[y_t \mid \theta_t = i] - \psi\left(\frac{1-\delta}{1-\delta^{T-t+1}} v\right)\right].$$

Define $\pi_{ij}^1 \equiv \pi_j$ and $\pi_{ij}^t \equiv \pi_{ij}$, for $t > 1$. It is easy to show that $\Pi_t(V) \leq \pi_{hl}^t \Pi_{t,l}^{FI}(V_l) + \pi_{hh}^t \Pi_{t,h}^{FI}(V_h)$. If $V_l \geq V_h$, both functions coincide since offering constant utility $\frac{1-\delta}{1-\delta^{T-t+1}} V_i$, following type $\theta_t = i$, is feasible (incentive constraints of relaxed problem do not bind).

1.2 Recursive representation

The following result is useful in studying the recursive structure of the relaxed problem.

Lemma 11. *If Π_t has a solution, (i) it is strictly concave and (ii) its solution is unique.*

Proof. Proof of (i). For any t, consider utility pairs $V^1, V^2 \in \mathcal{V}_t$, with $V^1 \neq V^2$, and $\alpha \in (0,1)$. For $k = 1, 2$, take optimal period t continuation UDMs $M_t^{u,k} \equiv \left\{\vartheta_\tau^k\right\}_{\tau=t}^T$ in problem $\Pi_t(V^i)$. The mechanism $M_t^{u,\alpha} = \left\{\alpha\vartheta_\tau^1 + (1-\alpha)\vartheta_\tau^2\right\}_{\tau=t}^T$ is feasible in $\Pi_t(V^\alpha)$ and, since the objective function in (23) is strictly concave, generates profits strictly above $\alpha\Pi_t(V^1) + (1-\alpha)\Pi_t(V^2)$. Since all incentive constraints as well as (24) are linear in utility flows, we have that $M_t^{u,\alpha} \equiv \left\{\vartheta_\tau^k\right\}_{\tau=t}^T$ is feasible in $\Pi_t(V^\alpha)$. Hence $\Pi_t(V^\alpha) > \alpha\Pi_t(V^1) + (1-\alpha)\Pi_t(V^2)$.

Proof of (ii). Follows similarly from strict concavity of objective function and linearity of constraints in problem $\Pi_t(\cdot)$. □

We extend the definition of ξ by defining, for any function $\vartheta : Y \mapsto u(\mathbb{R}_+)$,
$\xi(\vartheta, \theta) \equiv \sum_{y \in Y} p_\theta(y)\left[y - \psi\left[\vartheta(y)\right]\right]$. Define a period t policy as a any pair (ϑ, N) such that $\vartheta : Y \mapsto u(\mathbb{R}_+)$ and $N : \Phi \mapsto \mathcal{V}_{t+1}$, and the set of period t policies as $\mathcal{N}_t$. Finally, define

$$\Gamma_t \equiv \left\{ (V_l, V_h) \mid \begin{array}{c} \exists\,(\vartheta, N) \in \mathcal{N}_t \text{ such that} \\ V_h = \sum_{y\in Y} p_h(y)\left[\vartheta(y) + \pi_{hh} N_h(\phi(y)) + \pi_{hl} N_l(\phi(y))\right] \\ V_l \geq \sum_{y\in Y} p_l(y)\left[\vartheta(y) + \pi_{hh} N_h(\phi(y)) + \pi_{hl} N_l(\phi(y))\right] \\ V_l \in \frac{1-\delta^{T-t+1}}{1-\delta} u(\mathbb{R}_+) \end{array} \right\},$$

and the following optimization problem, choosing the optimal period t policy:

$$P_t(V) \equiv \begin{array}{c} \sup_{(\vartheta,N)\in\mathcal{N}_t} \pi_{hh}^t \left[\xi(\vartheta, h) + \delta \sum_{y\in Y} p_h(y)\,\Pi_{t+1}(N(\phi(y)))\right] + \pi_{hl}^t \Pi_{t,l}^{FI}(V_l) \\ \text{s.t.} \left\{ \begin{array}{c} V_l \geq \sum_{y\in Y} p_l(y)\left\{\vartheta(y) + \delta\left[\pi_{lh} N_h(\phi(y)) + \pi_{ll} N_l(\phi(y))\right]\right\} \\ V_h = \sum_{y\in Y} p_h(y)\left\{\vartheta(y) + \delta\left[\pi_{hh} N_h(\phi(y)) + \pi_{hl} N_l(\phi(y))\right]\right\} \end{array} \right. \end{array}, \quad (25)$$

with solution, whenever it exists, denoted as $\left(\vartheta_t^V, N_t^V\right)$.

Lemma 12. *For any $t = 1, ..., T$ and $V \in \mathcal{V}_t$, let $\{\vartheta_\tau^*\}_{\tau=t}^T$ be the solution to problem (23). The following hold, for any $t \leq T$:*

(i) (Full insurance following low type) if UDM $\{\vartheta_t^*\}_{\tau=t}^T$ solves $\Pi_t(V)$, then, for any $1 \leq \tau \leq T - t + 1$ and $y^{\tau-1} \in Y^{\tau-1}$:
$\vartheta_{t+\tau'}(\theta^{\tau'}, y^{\tau'}) = \vartheta_{t+\tau''}(\theta^{\tau''}, y^{\tau''})$ *if $\theta^{\tau'}, \theta^{\tau''} \succeq (h^{\tau-1}, l)$ and $y^{\tau'}, y^{\tau''} \succeq y^{\tau-1}$.*

(ii) Recursive utility set representation: $\mathcal{V}_t = \Gamma_t$,

(iii) $\Pi_t(\cdot)$ *satisfies the following recursive equation, i.e.,* $\Pi_t(\cdot) = P_t(\cdot)$,

(iv) Flow contract and utilities following high type: for any $y \in Y$, $V \in \mathcal{V}_t$ *and* $i \in \{l, h\}$, $\vartheta_t^*(h, y) = \vartheta_t^V(y)$,

$$\mathbb{E}\left[\sum_{\tau=t+1}^{T} \delta^{\tau-t-1}\vartheta\left([\theta]_t^{\tau}, [y]_t^{\tau}\right) \mid \theta^t = \left(h^t, i\right), y_t = y\right] = N_i^V(\phi(y)).$$

Proof. We start by proving (i). Consider problem $\Pi_t(V)$, $1 \le \tau \le T-t+1$, and $y^{\tau-1} \in Y^{\tau-1}$. Utility flows for histories $(\theta^{\tau'}, y^{\tau'})$ satisfying $\theta^{\tau'} \succeq (h^{\tau-1}, l)$ and $y^{\tau'} \succeq y^{\tau-1}$ only affect the objective function and incentive constraints, respectively, through

$$\mathbb{E}\left\{\sum_{\tau'=t+\tau-1}^{T} \delta^{\tau'-t}\left[y_{\tau'} - \psi\left[\vartheta_{\tau'}\left(\left[\theta^{\tau'}\right]_t^{\tau'}, \left[y^{\tau'}\right]_t^{\tau'}\right)\right]\right] \mid [\theta^T]_t^{t+\tau-1} = (h^{\tau-1}, l), [y^T]_t^{t+\tau-2} = y^{\tau-1}\right\}, \tag{26}$$

$$\text{and } \mathbb{E}\left\{\sum_{\tau'=t+\tau-1}^{T} \delta^{\tau'-t}\vartheta_{\tau'}\left(\left[\theta^{\tau'}\right]_t^{\tau'}, \left[y^{\tau'}\right]_t^{\tau'}\right) \mid [\theta^T]_t^{t+\tau-1} = (h^{\tau-1}, l), [y^T]_t^{t+\tau-2} = y^{\tau-1}\right\}. \tag{27}$$

Convexity of ψ implies that a constant utility flow, following type realizations $(h^{\tau-1}, l)$ and income realizations $y^{\tau-1}$ uniquely maximizes (26), subject to (27). Hence any solution to the profit maximization problem Π_t must satisfy (i), in which case (26) is equal to $\Pi_{t,l}^{FI}(V_l)$.

Hence, we can restrict the choice set in the profit minimization problem to the set $\bar{\mathcal{M}}_{t,OSIC}^u$, which is the subset of $\mathcal{M}_{t,OSIC}^u$ satisfying (i). As a consequence, the continuation mechanism following a low-type announcement is pinned down by the continuation utility of the consumer: it corresponds to the constant consumption amount that delivers such continuation utility. We now show (ii)-(iv).

Fix a period any t. Any OSIC mechanism in $\bar{\mathcal{M}}_{t,OSIC}^u$ can be described by the continuation utility obtained conditional on $\theta_t = l$, the period t utility flow generated to a consumer with $\theta_t = h$ and the continuation utility flows provided from period $t+1$ onwards for a consumer with $\theta_t = h$. This statement is formalized below.

Define $\bar{A}_t \equiv \mathbb{R} \times [u(\mathbb{R})]^Y \times \left[\mathcal{M}_{t+1,OSIC}^u\right]^{\Phi}$ as the set of triples $(V_l, \vartheta(\cdot), \bar{N}(\cdot))$ where, in period t with type announcement h and income realization $y \in Y$, $\vartheta(y)$ represents flow utility in period t and $\bar{N}(\phi(y)) \in \mathcal{M}_{t+1,OSIC}^u$ represents the continuation mechanism offered from period $t+1$ onwards. The flow utility in a continuation mechanism $\bar{N}(\phi(y))$, if history $(\theta^{\tau}, y^{\tau})$ are observed starting in period $t+1$ is denoted as $\bar{N}(\theta^{\tau}, y^{\tau}; \phi(y))$. So a one-to-one mapping exists between the set $\bar{\mathcal{M}}^u_{t,OSIC}$ and the set A_t defined as

$$\left\{ \left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right) \in \bar{A}_t \mid \begin{array}{c} \exists \left(V_l', V_h'\right) \in \mathbb{R}^2 \text{ s.t.} \\ V_i'\left(\phi\right) = \mathbb{E}\left[\sum_{\tau=t+1}^{T} \delta^{\tau-t-1} \bar{N}\left(\left[\theta^\tau, y^\tau\right]_{t+1}^{\tau}; \phi\right) \mid \theta_{t+1} = i\right], \\ V_l \geq \sum_{y \in Y} p_h\left(y\right)\left[\vartheta\left(y\right) + \delta(\pi_{lh} V_h'\left(\phi\left(y\right)\right) + \pi_{ll} V_l'\left(\phi\left(y\right)\right))\right], \\ V_l \in \frac{1-\delta^{T-t+1}}{1-\delta} u\left(\mathbb{R}_+\right). \end{array} \right\}. \tag{28}$$

The one-to-one mapping $a : A_t \mapsto \bar{\mathcal{M}}^u_{t,OSIC}$ assigns, for each $\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right) \in A_t$, the mechanism $a\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right) = \left\{\vartheta_\tau\right\}_{\tau=t}^{T}$ satisfying: $\vartheta_\tau\left(\theta^{\tau-t}, y^{\tau-t}\right) = \frac{1-\delta}{1-\delta^{T-t+1}} V_l$, for any $\left(\theta^{\tau-t}, y^{\tau-t}\right) \succeq \left(l, y\right)$ and any $y \in Y$; $\vartheta_t\left(h, y\right) = \vartheta\left(y\right)$ for any $y \in Y$; and for any $\tau \geq t+1$ and $\left(\theta^{\tau-t-1}, y^{\tau-t-1}\right)$, $\vartheta_\tau\left(\left(h, \theta^{\tau-1}\right), \left(y, y^{\tau-1}\right)\right) = \bar{N}\left(\theta^{\tau-1}, y^{\tau-1} \mid \phi\left(y\right)\right)$. The mechanism $a\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right)$ satisfies OSIC since: (i) the inequality in expression (28) implies that the period t one-shot incentive constraint is satisfied, and (ii) the one-shot incentive constraints for periods $\tau \geq t+1$ are guaranteed since the continuation mechanism $\bar{N}\left(\phi\left(y_t\right)\right)$, for any $y_t \in Y$, is contained in $\mathcal{M}^u_{t+1,OSIC}$. It is easy to show that, for any $M^u_t \in \bar{\mathcal{M}}^u_{t,OSIC}$, the vector $a^{-1}\left(M^u_t\right)$ is an element of A_t. Moreover, the set of utilities generated by mechanisms $a\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right)$, for all $\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right) \in A_t$ coincides with Γ_t, which implies (ii).

Now consider any $V \in \mathcal{V}_t$ and any mechanism $a\left(V_l, \vartheta\left(\cdot\right), \bar{N}\left(\cdot\right)\right) \in \bar{\mathcal{M}}^u_{t,OSIC}$ feasible in the problem defining $\Pi_t\left(V\right)$. Define $N = \left(N_l, N_h\right) : \Phi \mapsto \mathbb{R}^2$, for each $i \in \{l, h\}$, by

$$N_i\left(\phi\right) \equiv \mathbb{E}\left[\sum_{\tau=t+1}^{T} \delta^{\tau-t-1} \bar{N}\left(\left[\theta^\tau, y^\tau\right]_{t+1}^{\tau}; \phi\right) \mid \theta_{t+1} = i\right].$$

The new mechanism $a\left(V_l, \vartheta\left(\cdot\right), M^{*,N(\cdot)}_{t+1}\right)$, where $M^{*,N(\phi)}_{t+1} \in \mathcal{M}^u_{t+1,OSIC}$ is the optimal mechanism in the problem $\mathcal{P}_{t+1}\left(N_l\left(\phi\right), N_h\left(\phi\right)\right)$, for all $\phi \in \Phi$, is also in A_t, generates the same continuation utility in period t, for any $\theta_t \in \{l, h\}$, and generates strictly higher profits if $N^{*,N(\cdot)}_{t+1} \neq \bar{N}$. Hence, without loss of optimality in problem $\mathcal{P}_t\left(V\right)$, we can focus on mechanisms indexed by utility level V_l and mappings $\left(\vartheta, N\right) : Y \mapsto u\left(\mathbb{R}_+\right) \times \mathcal{V}_{t+1}$, which are given by $a\left(V_l, \vartheta\left(\cdot\right), M^{*,N(\cdot)}_{t+1}\right)$. From now on, we refer to such mechanisms via the triple $\left(V_l, \vartheta, N\right) \in u\left(\mathbb{R}_+\right) \times \left[u\left(\mathbb{R}_+\right)\right]^Y \times \left[\mathcal{V}_{t+1}\right]^\Phi$. The mechanism connected with $\left(V_l, \vartheta, N\right)$ satisfies OSIC and constraint (24) if, and only if, it satisfies

$$V_l \geq \sum_{y \in Y} p_l\left(y\right)\left\{\vartheta\left(y\right) + \delta\left[\pi_{hh} N_h\left(\phi\left(y\right)\right) + \pi_{hl} N_l\left(\phi\left(y\right)\right)\right]\right\}, \tag{29}$$

$$V_h = \sum_{y \in Y} p_h\left(y\right)\left\{\vartheta\left(y\right) + \delta\left[\pi_{hh} N_h\left(\phi\left(y\right)\right) + \pi_{hl} N_l\left(\phi\left(y\right)\right)\right]\right\}$$

and it generates profits

$$\pi_{hh}^{t}\left[\xi\left(\vartheta,h\right)+\delta\sum_{y\in Y}p_{h}\left(y\right)\Pi_{t+1}\left(N\left(\phi\left(y\right)\right)\right)\right]+\pi_{hl}^{t}\Pi_{t,l}^{FI}\left(V_{l}\right).$$

Hence, the maximal profit $\Pi_t(\cdot)$ must satisfy (25) and the profit maximizing continuation UDM must be generated by the solution to (25). This implies (iii) and (iv). □

1.3 Properties

We use notation $\partial_i^+\Pi_t(\cdot)$ and $\partial_i^-\Pi_t(\cdot)$ to represent the right- and left-derivatives of $\Pi_t(\cdot)$ with respect to V_i, for $i=l,h$. For any time t, continuation utility vector $V\in int(\mathcal{V}_t)$ and constant c, we denote the expression $0\in[\partial_i^+\Pi_t(V)+c,\partial_i^-\Pi_t(V)+c]$ simply by $0\in\partial_i\Pi_t(V)+c$.

Lemma 13. *For any $V\in int(\mathcal{V}_t)$ with $V_l\geq V_h$, $\Pi_t(\cdot)$ is differentiable at V and, using $u_i=V_i\left(\sum_{\tau=t}^{T}\delta^{\tau-t}\right)^{-1}$:*

$$\frac{\partial}{\partial V_i}\Pi_t\left(V\right)=\pi_{hi}^{t}\frac{d}{dV_i}\Pi_{t,i}^{FI}\left(V_i\right)=\pi_{hi}^{t}\psi'(u_i).$$

Proof. Since $\Pi_t(V)=\pi_{hh}^t\Pi_{t,h}^{FI}(V_h)+\pi_{hl}^t\Pi_{t,l}^{FI}(V_l)$, for $V_l\geq V_h$, the result is obvious for $V_l>V_h$. Now consider a pair $(V_0,V_0)\in\mathcal{V}_t$, with $V_0\equiv\frac{1-\delta^{T-t+1}}{1-\delta}u_0$ and $u_0>u(0)$. The optimal period t policy $\left(\vartheta^V,N^V\right)$ induces constant utility flow u_0, i.e., $\vartheta^V(y)=u_0$, for all $y\in Y$. Now for ε sufficiently small, define an alternative policy $\left(\tilde{\vartheta},N^V\right)$ with utility flow given by

$$\tilde{\vartheta}^{\varepsilon}\left(y\right)\equiv u_0+\varepsilon\frac{p_l\left(y\right)^{-1}-|Y|}{\sum_{y\in Y}\ell\left(y\right)^{-1}-|Y|}.$$

This alternative policy is feasible in problem $\mathcal{P}_t(V_0+\varepsilon,V_0)$, which implies

$$\Pi_t\left(V_0+\varepsilon,V_0\right)-\Pi_t\left(V_0,V_0\right)\geq\pi_{hh}^{t}\sum_{y\in Y}p_h\left(y\right)\left\{\psi\left(u_0\right)-\psi\left[\tilde{\vartheta}^{\varepsilon}\left(y\right)\right]\right\},$$

holding as an equality for $\varepsilon=0$. Since the right-hand side (RHS) is differentiable and the left-hand side is strictly concave, their derivative coincides. The derivative of the RHS at $\varepsilon=0$ is $\pi_{hh}^t\psi'(u_0)=\pi_{hh}^t\frac{d}{dv}\Pi_{t,h}^{FI}(V_0)$, which pins down $\frac{\partial}{\partial V_h}\Pi_t(V_0,V_0)$. A similar argument can be used to find $\frac{\partial}{\partial V_l}\Pi_t(V_0,V_0)$, with an ε-perturbation that is feasible in problem $\mathcal{P}_t(V_0,V_0+\varepsilon)$, for ε small. □

The following result shows what type of distortions arise in the profit maximizing mechanism and the dynamic behavior of continuation utilities.

Lemma 14. *For any $V \in int(\mathcal{V}_t)$, the solution $\left(\vartheta^V, N^V\right)$ of (25) satisfies: there exists multipliers $\lambda \geq 0$ and $\mu > 0$ such that*

(i) Multiplier signal: $\lambda > 0$ and both constraints in (25) bind if, and only if, $V_h > V_l$,

(ii) Within-period distortions: ϑ_t^V satisfies

$$-\psi'\left(\vartheta^V(y)\right) + \mu - \lambda\ell(y) \begin{cases} \leq 0 & \vartheta^V(y) = u(0), \\ = 0 & , \text{ if } \vartheta^V(y) > u(0). \end{cases}$$

(iii) Continuation utility rewards high types: if $V_h > V_l$, $t < T$, then $N_h^V(\phi) > N_l^V(\phi)$, for any $\phi \in \Phi$ such that $N_l^V(\phi), N_h^V(\phi) > \frac{1-\delta^{T-t+1}}{1-\delta}u_0$,

(iv) Continuation utility signal rewards: for any $\phi, \phi' \in \Phi$

$$\ell(\phi') \leq \ell(\phi) \implies \pi_{ll}N_l^V(\phi') + \pi_{lh}N_h^V(\phi') \geq \pi_{ll}N_l^V(\phi) + \pi_{lh}N_h^V(\phi),$$

with this inequality holding strictly if $V_h > V_l$ and $N^V(\phi) \in int(\mathcal{V}_{t+1})$.

Proof. Consider arbitrary period t and $V \in int(\mathcal{V}_t)$.

Step 1. There exists period t policy $(\vartheta, N) \in \mathcal{N}_t$ feasible in problem $\mathcal{P}_t(V)$ such that inequality constraint in (25) holds strictly: just consider a feasible policy in problem $\mathcal{P}_t(V_l - \varepsilon, V_h)$, for $\varepsilon > 0$ small.

Step 2. The optimization problem $\mathcal{P}_t(V)$ has a concave objective, convex choice set $\mathcal{N}_t$ and linear constraints. Step 1 implies that a feasible policy $\left(\vartheta^V, N^V\right) \in \mathcal{N}_t$ for problem $\mathcal{P}_t(V)$ is optimal if, and only if, there exists multipliers $\lambda \geq 0$ and $\mu \in \mathbb{R}$ such that $\left[\left(\vartheta^V, N^V\right), (\lambda)\right] \in \mathcal{N}_t \times \mathbb{R}_+$ form a saddle point of the Lagrangian

$$\xi(\vartheta, h) + \sum_{y \in Y} p_h(y)\vartheta(y)[\mu - \lambda\ell(y)] + \lambda V_l - \mu V_h$$
$$+ \sum_{\phi \in \Phi} p_h(\phi) \left\{ \begin{array}{c} \Pi_{t+1}(N(\phi(y))) + [\mu\pi_{hh} - \lambda\ell(\phi)\pi_{lh}] N_h(\phi(y)) \\ + [\mu\pi_{hl} - \lambda\ell(\phi)\pi_{ll}] N_l(\phi(y)) \end{array} \right\}. \quad (30)$$

Moreover, the necessary condition for optimality of $N(\cdot)$ is

$$0 \in \partial_i \Pi_{t+1}(N(\phi(y))) + \mu\pi_{hi} - \lambda\ell(\phi)\pi_{li}, \quad (31)$$

while (ii) is a necessary condition for optimality of $\vartheta(\cdot)$. Since $\partial_h^- \Pi_{t+1}(N(\phi(y))) < 0$, (31) implies $\mu > 0$.

Step 3 (item i). If $\lambda = 0$, (30) becomes the Lagrangian of the problem $\bar{\mathcal{P}}_t(V)$ where the period t incentive constraint is ignored, which has as unique solution constant flow utility equal to $\frac{1-\delta}{1-\delta^{T-t+1}}V_i$, for $\hat{\theta}_t = i$, only satisfying the period t one-shot incentive constraint if $V_h \leq V_l$. Alternatively, if $V_l \geq V_h$, the optimal mechanism in $\mathcal{P}_t(V)$ involves constant utilities that do not depend on income y, which is only optimal in (30) if $\lambda = 0$.

Step 4 (item ii). Property (ii) is equivalent to local optimality of ϑ^V in (30).

Step 6 (item iii). If $V_l < V_h$ and $N_h^V(\phi_0) \leq N_l^V(\phi_0)$ for some $\phi_0 \in \Phi$, a contradiction follows. Necessary condition (31) implies

$$\partial_h^+ \Pi_{t+1}\left(N^V(\phi_0)\right) + \left[\mu\pi_{hh} - \lambda\ell(\phi_0)\pi_{lh}\right] \leq 0 \leq \partial_l^- \Pi_{t+1}\left(N^V(\phi_0)\right) + \left[\mu\pi_{hl} - \lambda\ell(\phi_0)\pi_{ll}\right]$$

which, using $\lambda > 0$, gives us

$$\frac{\partial_h^+ \Pi_{t+1}\left(N^V(\phi_0)\right)}{\pi_{hh}} \leq \lambda\ell(\phi_0)\frac{\pi_{lh}}{\pi_{hh}} - \mu < \lambda\ell(\phi_0)\frac{\pi_{ll}}{\pi_{hl}} - \mu \leq \frac{\partial_l^- \Pi_{t+1}\left(N^V(\phi_0)\right)}{\pi_{hl}}. \tag{32}$$

Finally, $N_h^V(\phi_0) \leq N_l^V(\phi_0)$ and Lemma 13 imply that Π_{t+1} is differentiable at V and

$$\frac{\partial_i \Pi_{t+1}\left(N^V(\phi_0)\right)}{\pi_{hi}} = \frac{d}{dV_i}\Pi_{t+1,i}^{FI}\left(N_i^V(\phi_0)\right) = \frac{d}{dV_l}\Pi_{t+1,l}^{FI}\left(N_i^V(\phi_0)\right)$$

but, given convexity of $\Pi_{t+1,l}^{FI}(\cdot)$, we have $\frac{\partial_h \Pi_{t+1}\left(N^V(\phi_0)\right)}{\pi_{hh}} > \frac{\partial_l \Pi_{t+1}\left(N^V(\phi_0)\right)}{\pi_{hl}}$, contradicting (32).

Step 7 (item iv). For any $\phi \in \Phi$, equation (31) implies

$$\begin{bmatrix} \lambda\ell(\phi)\pi_{ll} - \mu\pi_{hl} \\ \lambda\ell(\phi)\pi_{lh} - \mu\pi_{hh} \end{bmatrix} \in \partial\Pi_{t+1}(N^V(\phi(y)))$$

Considering any two $\phi, \phi' \in \Phi$, convexity of $\Pi_{t+1}(\cdot)$ implies:

$$\Pi_{t+1}(N^V(\phi')) - \Pi_{t+1}(N^V(\phi)) \leq \begin{bmatrix} \lambda\ell(\phi)\pi_{ll} - \mu\pi_{hl} \\ \lambda\ell(\phi)\pi_{lh} - \mu\pi_{hh} \end{bmatrix}^T [N^V(\phi') - N^V(\phi)]$$

$$\Pi_{t+1}(N^V(\phi)) - \Pi_{t+1}(N^V(\phi')) \leq \begin{bmatrix} \lambda\ell(\phi')\pi_{ll} - \mu\pi_{hl} \\ \lambda\ell(\phi')\pi_{lh} - \mu\pi_{hh} \end{bmatrix}^T [N^V(\phi) - N^V(\phi')]$$

Item (iv) follows from summing up both equations, which gives us:

$$0 \leq \lambda[\ell(\phi) - \ell(\phi')] \begin{bmatrix} \pi_{ll} \\ \pi_{lh} \end{bmatrix}^T [N^V(\phi') - N^V(\phi)].$$

□

2 Proof of Lemma 3

Proof. Uniqueness follows from Lemma 11. The remaining statements are proved in the order: $ii \to iv \to i \to iii \to v$.

Statement (ii) follows directly from Lemma 12-(i), which has $\Pi_1 = \Pi^*$ as a special case.

All other properties are derived from Lemma 14. For any $t = 1, \ldots, T$ and $\eta^{t-1} = (h^{t-1}, \phi^{t-1})$, we have that $z_t(y \mid \eta^{t-1}, h) = \vartheta(y)$ and $V_{t+1}(\eta^{t-1}, \phi_t, h, i) = N_i(\phi_t)$, for $i = l, h$; where (ϑ, N) is the solution to problem $P_t\left(V_t(\eta^{t-1}, l), V_t(\eta^{t-1}, h)\right)$.

Proof of (iv). Follows from $V_l < V_h$ and Lemma 14-(iii).

Proof of (i). For any history $\eta^{t-1} = (h^{t-1}, \phi^{t-1})$, the result follows from (iv), which states that the continuation utility following $\theta_t = h$ is strictly higher than that following $\theta_t = l$, and Lemma 14-(i), which implies that the within-period t upward incentive constraint binds as a result.

Let $\mu_{t-1}(\phi^{t-1})$ and $\lambda_{t-1}(\phi^{t-1})$ be the Lagrange multipliers of the problem $P_t\left(V_t(\eta^{t-1}, l), V_t(\eta^{t-1}, h)\right)$, with $\eta^{t-1} = (h^{t-1}, \phi^{t-1})$.

Proof of (iii). Follows from Lemma 14-(ii), using the following relationship to the solution $\vartheta(\cdot)$ of problem $P_t\left(V_t(\eta^{t-1}, l), V_t(\eta^{t-1}, h)\right)$:

$$\psi'(\vartheta(y_t)) = \frac{1}{u'\left(z_t(y_t \mid \eta^{t-1}, h)\right)}.$$

Proof of (v). Follows directly from Lemma 14-(iv). □

3 Results on the auxiliary problem

For simplicity, I write the auxiliary problem in terms of utility levels:

$$\chi\left(\nu, \Delta\right) = \inf_{x:Y \to u(\mathbb{R}_+)} \sum_{y} p_h\left(y\right) \psi\left[x\left(y\right)\right],$$

subject to $\sum_y p_h\left(y\right) x\left(y\right) = \nu$, and $\sum_y p_l\left(y\right) x\left(y\right) = \nu - \Delta$.
The following statement provides important properties of cost function χ used in our analysis.

Lemma 15. *The problem $\mathcal{P}^A$ satisfies the following:*
(i) It has a unique solution,
(ii) $\chi(\cdot)$ is strictly convex,
moreover, if $x\left(\cdot\right) \in int\left[u\left(\mathbb{R}_+\right)\right]^Y$ solves $\mathcal{P}^A$, then:
(iii) $\chi\left(\nu, \Delta\right)$ is twice continuously differentiable in an open neighborhood of (ν, Δ),
(iv) $sign\left(\frac{\partial \chi(\nu,\Delta)}{\partial \Delta}\right) = sign\left(\Delta\right)$,

(v) cross derivative sign:

$$sign\left(\frac{\partial^2\chi}{\partial\nu\partial\Delta}\right) = \begin{cases} sign\left(\Delta\right), \ if\ \psi''' > 0 \\ -sign\left(\Delta\right), \ if\ \psi''' < 0 \\ = 0, \ if\ \psi''' = 0, \end{cases}$$

(vi) $\psi'(x(y)) = \chi_\nu(\nu,\Delta) + \chi_\Delta(\nu,\Delta)\left[1-\ell(y)\right]$.

Proof. Existence of solution follows from the fact that $\left\{x \in \left[u\left(\mathbb{R}_+\right)\right]^Y \mid \sum_y p_h\left(y\right)\psi\left[x\left(y\right)\right] \leq K\right\}$ is compact, for any $K \in \mathbb{R}_+$. Uniqueness and convexity (items i-ii) follows from the strict convexity of the objective function and linearity of the constraints in (x,ν,Δ).

The following are necessary and sufficient conditions for $x\left(\cdot\right) \in int\left[u\left(\mathbb{R}_+\right)\right]^Y$ to be interior are: $\exists\lambda,\mu \in \mathbb{R}$ such that, for all $y \in Y$,

$$\psi'\left(x\left(y\right)\right) - \lambda + \mu\ell(y) = 0, \tag{33}$$

$$\sum_{y\in Y} p_h\left(y\right)x\left(y\right) = \nu \text{ and } \sum_{y\in Y} p_l\left(y\right)x\left(y\right) = \nu - \Delta. \tag{34}$$

Consider $\left\{y_i\right\}_{i\in I}$ an ordering of Y such that $\left\{\ell(y_i)\right\}_{i\in I}$ is increasing. Then distributions $\left\{p_h\left(y_i\right)\right\}_{i\in I}$ and $\left\{p_l\left(y_i\right)\right\}_{i\in I}$ are ordered in terms of the monotone likelihood ratio property (MLRP). It then follows that $\left\{x\left(y_i\right)\right\}_{i\in I}$ is decreasing (increasing) if $\mu > 0$ ($\mu < 0$), which implies that Δ must be strictly positive (negative). As a consequence, $sign\left(\mu\right) = sign\left(\Delta\right)$ (item iv).

If $\left(\lambda,\mu,x\left(\cdot\right)\right)$ solve $(33)-(34)$, then by the implicit function theorem the system has a unique continuously differentiable solution $\left(\lambda^{\nu',\Delta'},\mu^{\nu',\Delta'},x\left(\cdot\mid\nu',\Delta'\right)\right)$ for (ν',Δ') in an open neighborhood of (ν,Δ). Therefore χ is continuously differentiable at (ν,Δ), and its derivative is given by

$$\begin{bmatrix} \frac{\partial}{\partial\nu}\chi\left(\nu,\Delta\right) \\ \frac{\partial}{\partial\Delta}\chi\left(\nu,\Delta\right) \end{bmatrix} = \begin{bmatrix} \lambda^{\nu,\Delta} - \mu^{\nu,\Delta} \\ \mu^{\nu,\Delta} \end{bmatrix}. \tag{35}$$

Continuous differentiability of $\left(\lambda^{\nu,\Delta},\mu^{\nu,\Delta}\right)$ implies that $\chi\left(\cdot\right)$ is twice continuously differentiable at (ν,Δ) (item iii).

Finally, simple differentiation implies

$$\frac{\partial^2\chi\left(\nu,\Delta\right)}{\partial\nu\partial\Delta} = \frac{\sum_y \frac{p_l(y)}{\psi''(x(y))} - \sum_y \frac{p_h(y)}{\psi''(x(y))}}{\left(\sum_y \frac{p_h(y)}{\psi''(x(y))}\right)\left(\sum_y \frac{[p_l(y)]^2}{p_h(y)\psi''(x(y))}\right) + \left(\sum_y \frac{p_l(y)}{\psi''(x(y))}\right)^2}.$$

Now assume $\psi''' > 0$, $\left\{\frac{1}{\psi''(x(y_i))}\right\}_{i\in I}$ is increasing (decreasing) in i if, only if, $\Delta > 0$ ($\Delta < 0$).

Since $\{p_h(y_i)\}_{i\in I}$ and $\{p_l(y_i)\}_{i\in I}$ are MLRP ordered, we have that

$$sign\left(\frac{\partial^2\chi(\nu,\Delta)}{\partial\nu\partial\Delta}\right) = sign(\Delta).$$

An analogous argument works if $\psi''' = 0$ or $\psi''' < 0$ (item v). (33) and (35) imply (vi). □

Lemma 16. *The solution to $\mathcal{P}^A$ for the pair (ν,Δ) satisfies*

$$\frac{1}{u'(\zeta(y))} = \chi_\nu(\nu,\Delta) + \chi_\Delta(\nu,\Delta)\left[1-\ell(y)\right].$$

Proof. Follows from Lemma 15, with the solution in terms of utility flows and consumption being connected via $\frac{1}{u'(\zeta(y))} = \psi'(x(y))$. □

The following statement links variation in marginal cost of utility and distortions to the levels of utility and distortions and will be instrumental in the analysis of Subsection 6.1.

Lemma 17. *Suppose $\chi(\cdot)$ is strictly convex and $\frac{\partial^2\chi}{\partial\nu\partial\Delta} > 0$, then*

$$\left\{\begin{array}{c}\chi_\nu(\nu,\Delta) \geq \chi_\nu(\nu',\Delta'),\\ \chi_\Delta(\nu,\Delta) \leq \chi_\Delta(\nu',\Delta')\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\nu \geq \nu',\\ \Delta \leq \Delta'\end{array}\right\}.$$

If the left conditions hold with strict inequalities, then the implications are also strict.

Proof. Since χ is twice continuously differentiable, the following holds:

$$\chi_\nu(\nu,\Delta) - \chi_\nu(\nu',\Delta') = (\nu-\nu')\int_0^1 \chi_{vv}(\iota(\alpha))\,d\alpha + (\Delta-\Delta')\int_0^1 \chi_{\nu\Delta}(\iota(\alpha))\,d\alpha \geq 0,$$

$$\chi_\Delta(\nu,\Delta) - \chi_\Delta(\nu',\Delta') = (\nu-\nu')\int_0^1 \chi_{\nu\Delta}(\iota(\alpha))\,d\alpha + (\Delta-\Delta')\int_0^1 \chi_{\Delta\Delta}(\iota(\alpha))\,d\alpha \leq 0,$$

where $\iota(\alpha) = \alpha(\nu,\Delta) + (1-\alpha)(\nu',\Delta')$, for $\alpha \in [0,1]$.

Using $\frac{\partial^2\chi}{\partial\nu\partial\Delta} > 0$, these imply

$$(\Delta-\Delta')\left[\frac{\int_0^1 \chi_{\nu\Delta}(\iota(\alpha))\,d\alpha}{\int_0^1 \chi_{vv}(\iota(\alpha))\,d\alpha} - \frac{\int_0^1 \chi_{\Delta\Delta}(\iota(\alpha))\,d\alpha}{\int_0^1 \chi_{\nu\Delta}(\iota(\alpha))\,d\alpha}\right] \geq 0.$$

However, convexity of $\chi(\cdot)$ second order continuous differentiability, implies that the function Γ defined over a neighborhood of $(0,0)$, given by $(v_0,\Delta_0) \mapsto \int_0^1 \chi(f(\alpha)+(v_0,\Delta_0))\,d\alpha$ is also convex and twice continuously differentiable. Convexity implies that

$$|\Gamma''(0,0)| = \left(\int_0^1 \chi_{vv}(\iota(\alpha))\, d\alpha\right)\left(\int_0^1 \chi_{\Delta\Delta}(\iota(\alpha))\, d\alpha\right) - \left(\int_0^1 \chi_{\nu\Delta}(\iota(\alpha))\, d\alpha\right)^2 > 0.$$

This implies that $\Delta \leq \Delta'$, which together with $\chi_\nu(\nu,\Delta) \geq \chi_\nu(\nu',\Delta')$ implies $\nu \geq \nu'$. Moreover, if the left inequalities in the Lemma hold strictly, we have that $\Delta < \Delta'$ and $\nu > \nu'$. □

Proof of Lemma 8

Proof. The equivalence of (i) and (iii) follows from Lemma 15-(v).

The second order derivative of $\psi = u^{-1}$ is given by

$$\psi''(x) = \left(-\frac{u''(\psi(x))}{u'(\psi(x))}\right)\frac{1}{u'(\psi(x))^2}.$$

The absolute risk aversion of utility $u(\cdot)$ at $c \in \mathbb{R}_+$ is given by $r_u(c) \equiv -\frac{u''(c)}{u'(c)}$. The equivalence between (ii) and (iii) follows from direct derivation: using $x = u(c)$,

$$\psi'''(x) = \frac{r_u'(x)\, u'(c) + 2\,[r_u(x)]^2}{[u'(c)]^5}.$$

□

4 Utility and distortion dynamics

In this section, I focus throughout on a particular $t = 1, \ldots, T-1$ and $\phi^{t-1} \in \Phi^{t-1}$ and study the problem of reallocating both flow utilities and distortions across periods t and $t+1$, following series of announcements $\hat{\theta}^t = h^t$ and signals ϕ^{t-1}. For notational brevity, I will omit the dependence of distortions and flow utilities on ϕ^{t-1}, denoting $\nu_t(\phi^{t-1})$ and $\nu_{t+1}(\phi^{t-1},\phi)$ simply as ν_t and $\nu_{t+1}(\phi)$, for example.

Define problem $\mathcal{P}^I$

$$\min_{(\nu,\Delta,(\nu'(\phi),\Delta'(\phi),\nu^l(\phi))_{\phi\in\Phi})\in\bar{A}} \chi(\nu,\Delta) + \delta\sum_{\phi\in\Phi} p_h(\phi)\left[\pi_{hh}\chi(\nu'(\phi),\Delta'(\phi)) + \pi_{hl}\chi(\nu^l(\phi),0)\right]$$

subject to:

$$\nu + \delta\sum_{\phi\in\Phi} p_h(\phi)\left[\pi_{hh}\nu'(\phi) + \pi_{hl}\nu^l(\phi)\right] = V_t(h^{t-1},h), \tag{36}$$

$$\nu - \Delta + \delta\sum_{\phi\in\Phi} p_l(\phi)\left[\pi_{lh}\nu'(\phi) + \pi_{ll}\nu^l(\phi)\right] = V_t(h^{t-1},l), \tag{37}$$

$$\nu'(\phi) - \Delta'(\phi) + \delta \sum_{\phi' \in \Phi} p_l(\phi') \left[\pi_{lh} V_{t+2}(h^{t+1}, (\phi, \phi'), h) + \pi_{ll} V_{t+2}(h^{t+1}, (\phi, \phi'), l)\right] =$$

$$\nu^l(\phi) + \sum_{t'=t+2}^{T} \delta^{t'-t-1} u(c_{t+1}(\phi)). \quad (38)$$

With the set $\bar{A}$ defined as

$$\bar{A} \equiv \left\{ (\nu, \Delta, (\nu'(\phi), \Delta'(\phi), \nu^l(\phi))_{\phi \in \Phi}) \mid (\nu, \Delta), (\nu'(\phi), \Delta'(\phi)), (\nu^l(\phi), 0) \in A, \forall \phi \in \Phi \right\}.$$

Lemma 18. *The vector* $\left(\nu_t, \Delta_t, \left(\nu_{t+1}(\phi), \Delta_{t+1}(\phi), \nu_{t+1}^l(\phi)\right)_{\phi \in \Phi}\right)$ *solves problem* $\mathcal{P}^I$.

Proof. First notice that, from Proposition 1, vector $\left(\nu_t, \Delta_t, \left(\nu_{t+1}(\phi), \Delta_{t+1}(\phi), \nu_{t+1}^l(\phi)\right)_{\phi \in \Phi}\right)$ is feasible in $\mathcal{P}^I$.

For any $(\nu, \Delta, (\nu'(\phi), \Delta'(\phi), \nu^l)_{\phi \in \Phi}) \in \bar{A}$, consider a new mechanism that is identical to optimal mechanism M except for changing: (i) $z_t(h^{t-1}, h)$ to $\zeta(\nu, \Delta)$, (ii) $z_{t+1}(h^t, \phi, h)$ to $\zeta(\nu'(\phi), \Delta'(\phi))$, (iii) consumption in $t+1$ following signals (ϕ^{t-1}, ϕ) and announcements (h^t, l) to $\psi(\nu'(\phi))$.

All incentive constraints in relaxed problem for announcements in $t' \geq t+2$ are unaffected. All incentive constraints in relaxed problem for announcements in $t' \leq t-1$ are unaffected since (36) guarantees that the continuation payoff of the consumer at the start of period t is unchanged.

The period t incentive constraint, following ϕ^{t-1}, is guaranteed by (37), while all period $t+1$ incentive constraints are guaranteed by (38).

Finally, the mechanism modification only affects the firm's expected discounted profits via the expression in the objective function of problem $\mathcal{P}^I$. Optimality of M implies the result. □

Proof of Proposition 2. Using equations (36)-(38), we can simplify problem $\mathcal{P}^I$ to one where only flow utilities in period $t+1$ are chosen by finding expressions for distortions $(\Delta, (\Delta'(\phi))_{\phi \in \Phi})$ and period t flow utility ν, in terms of $(\nu'(\phi))_{\phi \in \Phi}$.

By assumption, the solution of this problem is interior and hence the following local optimality conditions must be satisfied:

$$\chi_v^1 + \chi_\Delta^1 \left[1 - \frac{\pi_{lh}}{\pi_{hh}} \ell(\phi)\right] = \chi_v^2(\phi) + \chi_\Delta^2(\phi), \quad (39)$$

$$\chi_v^1 + \chi_\Delta^1 \left[1 - \frac{\pi_{ll}}{\pi_{hl}} \ell(\phi)\right] = \chi_v^{2,l}(\phi) - \frac{\pi_{hh}}{\pi_{hl}} \chi_\Delta^2(\phi), \quad (40)$$

where $\chi_k^1 \equiv \chi_k(\nu_t, \Delta_t)$, $\chi_k^2(\phi) \equiv \chi_k(\nu_{t+1}(\phi), \Delta_{t+1}(\phi))$, for $k \in \{\nu, \Delta\}$, and $\chi_v^{2,l} \equiv \chi_v(\nu_{t+1}^l(\phi))$.

Multiplying (39) by $\pi_{hh} p_h(\phi)$ and (40) by $\pi_{hl} p_h(\phi)$, adding both equations, and finally summing across signals $\phi \in \Phi$ gives us (13). Subtracting (39) from 40, multiplying the result by $p_h(\phi)$ and summing over $\phi \in \Phi$ gives us (14). □

4.1 Realization-independent mechanisms

For brevity, we now define $\chi_k^t \equiv \chi_k(\nu_t, \Delta_t)$ and $\chi_\nu^{t,l} \equiv \chi_\nu(\nu_t^l, 0)$ for for $k \in \{\nu, \Delta\}$ and $t = 1, \ldots, T$. We also introduce notation $V_t^d \equiv V_t(h^{t-1}, h) - V_t(h^{t-1}, l)$. From the binding period t incentive constraint, it satisfies (using $V_{T+1}^d = 0$ if $T < \infty$)

$$V_t^d = \Delta_t + \delta(\pi_{hh} - \pi_{lh}) V_{t+1}^d. \tag{41}$$

Lemma 19. *For any $t = 2, \ldots, T$, if $V_h > V_l$, the optimal mechanism is interior and satisfies $\chi_\nu^t > \chi_\nu^{t,l}$, then it must satisfy: $\nu_{t-1} < \nu_t$ and $\Delta_{t-1} > \Delta_t$.*

Proof. The intertemporal optimality condition 15 implies that $\chi_\nu^t > \chi_\nu^{t-1} > \chi_\nu^{t,l}$ while, using $\chi_\nu^t > \chi_\nu^{t,l}$ and optimality condition (16), we have that $\chi_\Delta^{t-1} < \chi_\Delta^t$. Hence, we have that $\chi_\nu(\nu_t, \Delta_t) > \chi_\nu(\nu_{t-1}, \Delta_{t-1})$ and $\chi_\Delta(\nu_t, \Delta_t) < \chi_\Delta(\nu_{t-1}, \Delta_{t-1})$. Which, using Lemma 17, imply the result. □

Lemma 20. *For any $t = 2, \ldots, T$, if $V_h > V_l$ then an interior optimal mechanism satisfies: (i) $\chi_\nu^t > \chi_\nu^{t,l}$, and (ii) $V_{t-1}^d > V_t^d$.*

Proof. We prove this statement by induction. First, consider $t = T$. In this case we have that $\Delta_T = \nu_T - \nu_T^l$, and, since Proposition 1 guarantees that $\Delta_T > 0$, supermodularity implies (i) as:

$$\chi_\nu(\nu_T, \Delta_T) \geq \chi_\nu(\nu_T, 0) > \chi_\nu(\nu_T^l).$$

Lemma 19 then implies that $\Delta_{T-1} > \Delta_T$.

Property (ii) follows since $V_T^d = \Delta_t$ and $V_{T-1}^d = \Delta_{T-1} + \delta(\pi_{hh} - \pi_{lh})\Delta_T$.

Now suppose that properties (i)-(ii) hold for all $t' = t+1, \ldots, T$.

We start by showing that property (i) holds. The continuation utility of a high-type consumer and low-type satisfy, respectively, satisfies

$$V_t(h^{t-1}, h) = \nu_t + \delta\left[\pi_{hh} V_{t+1}(h^t, h) + \pi_{hl} V_{t+1}(h^t, l)\right], \text{ and } V_t(h^{t-1}, l) = \sum_{\tau=t}^{T} \delta^{\tau-t} \nu_t^l. \tag{42}$$

Substituting (42) into (41) gives us the following, after some manipulation:

$$\Delta_t = (\nu_t - \nu_t^l) + \sum_{\tau>t} \delta^{\tau-t}(\nu_{t+1}^l - \nu_t^l) + \delta\pi_{lh}V_{t+1}^d \tag{43}$$

Suppose, by way of contradiction, that $\chi_\nu^t < \chi_\nu^{t,l}$. Given supermodularity of χ, this inequality requires

$$\nu_t^l > \nu_t. \tag{44}$$

Also, from our inductive hypothesis, $\chi_\nu^{t+1} > \chi_\nu^{t+1,l}$, which together with (15) implies that

$$\chi_\nu^{t,l} > \chi_\nu^t = \pi_{hh}\chi_\nu^{t+1} + \pi_{hl}\chi_\nu^{t+1,l} > \chi_\nu^{t+1,l} \implies \nu_t^l > \nu_{t+1}^l. \tag{45}$$

Combining (43), (44) and (45) we have $\Delta_t < \delta\pi_{lh}V_{t+1}^d$ and, using (41) once again, we have that $V_t^d \leq \delta\pi_{hh}V_{t+1}^d$, which contradicts property (ii), which holds at period $t+1$ by our inductive assumption.

We now prove property (ii). Since property (i) holds for any $t' \geq t$, Lemma 19 implies that $\Delta_{t'-1} > \Delta_{t'}$ for all $t \leq t' \leq T$. Now notice that $V_t^d = \sum_{\tau=t}^T [\delta(\pi_{hh} - \pi_{lh})]^{\tau-t} \Delta_\tau$ and hence we have

$$V_{t-1}^d - V_t^d = \sum_{s=0}^{T-t} [\delta(\pi_{hh} - \pi_{lh})]^s (\Delta_{t+s-1} - \Delta_{t+s}) + [\delta(\pi_{hh} - \pi_{lh})]^{T-t} \Delta_T,$$

which is strictly positive since the series $\{\Delta_\tau\}_{\tau=t-1}^T$ is strictly increasing. □

Proof of Proposition 3.
Follows directly from Lemmas 19 and 20 □

5 Competitive analysis

Denote the set of firms is $\mathcal{F}$. Let $V^E = (V_l^E, V_h^E)$ denote the equilibrium utility level of the consumer conditional on her initial type, and $V^F = (V_l^F, V_h^F)$ denote the vector describing the utility level obtained by the consumer when accepting the equilibrium offer of firm $F \in \mathcal{F}$. Finally, I denote the equilibrium profit of firm F as Π_F^*.

Lemma 21. *Any pure strategy PBE has outcome M^{V^*}.*

Proof. The proof is divided into four parts.

I) $\Pi(V^E) = 0$ and firms make zero profits in equilibrium.

Optimality of the consumer's acceptance strategy implies that, for $F \in \mathcal{F}$ and $i = l, h$, the following hold: (i) $V_i^F \leq V_i^E$, and (ii) $V_i^F = V_i^E$ if j's offer is accepted with positive probability by type i.

Hence, if firm F's offer is accepted by consumer with type i with positive probability, its continuation profits are at most $\Pi_i(V_i^E, V_{i'}^F) \leq \Pi_i(V_i^E, V_{i'}^E)$, as increasing the utility of type $i' \neq i$ increases the profit opportunities from the consumer with type i. This means that the total profits obtained by firms is at most $\sum_{i=l,h} \pi_i \Pi_i(V^E) = \Pi(V^E)$. However, by offering mechanism $M^{V'}$ with $V' = V^E + (\epsilon, \epsilon)$, for $\epsilon > 0$ sufficiently small, each firm can guarantee profits $\Pi(V')$. In equilibrium, the offer made by each firm must dominate $M^{V'}$. Since profit function Π is continuous, we have $\lim_{\epsilon \to 0} \Pi(V^E + (\epsilon, \epsilon)) = \Pi(V^E)$. Combining the two implications, we have $\Pi(V^E) \geq \sum_{F \in \mathcal{F}} \Pi_F^* \geq |\mathcal{F}| \, \Pi(V^E)$. This implies that $\Pi(V^E) = 0$ and that firms make zero profits.

II) $V_i^E \geq V_l^{FI}$, for $i = l, h$.

First suppose that, by way of contradiction, $V_l^E < V_l^{FI}$. In this case any firm could guarantee positive profits by offering a mechanism with non-contingent constant consumption satisfying

$$u(\bar{c}) \sum_{t=1}^{T} \delta^{t-1} = V_l^E + \epsilon,$$

for $\epsilon > 0$ sufficiently small, since it makes positive profits from the l-type consumer and, if the h-type consumer were to choose this contract, it would also generate positive profits since the discounted average income of the h-type consumer is strictly higher than that of the l-type consumer. Finally, if $V_h^E < V_l^{FI}$, a similar profitable deviation exists.

III) $\Pi_h(V^E) \leq 0$.

Suppose, by way of contradiction, that $\Pi_h(V^E) > 0$. This implies that $V_h^E < V_h^{FI}$ and hence concavity of the feasible utility set $\mathcal{V}$ implies that, for $\epsilon > 0$ sufficiently small, $(V_l^E, V_h^E + \epsilon) \in \mathcal{V}$.

Also, we know that $V_l^E \geq V_l^{FI} > \sum_{t=1}^{T} \delta^{t-1} u(0)$ and hence incentive compatibility implies that both types' equilibrium utility are strictly above $\sum_{t=1}^{T} \delta^{t-1} u(0)$. As a consequence, an incentive compatible mechanism in which the utility of both consumers is reduced exists, i.e., there exists $V' \in \mathcal{V}$ such that $V_i' < V_i^E$, for $i = l, h$.

The existence of feasible utility vectors $(V_l^E, V_h^E + \epsilon)$, for $\epsilon > 0$ and V', together with convexity of $\mathcal{V}$, implies that: for $\varepsilon > 0$ sufficiently small, there exists utility vector $\tilde{V} \in \mathcal{V}$ such that $V_l^E - \varepsilon < \tilde{V}_l < V_l^E$ and $V_h^E + \varepsilon > \tilde{V}_h > V_h^E$. Considering ε sufficiently small, a deviation to offer $M^{\tilde{V}}$ leads to profits approximately equal to $\pi_h \Pi_h(V^E) > 0$, which contradicts I).

IV) Points II and III imply that total profits are non-positive, conditional on any realization of the consumer's initial type. Combined with I, it implies that $\Pi_i(V^E) = 0$, for both $i = l, h$. Using II, we must have $V_l^E = V_l^{FI}$. Finally, since $\Pi_h(V_l^E, \cdot)$ is strictly decreasing in $(V_l^{FI}, \infty) \cap \mathcal{V}$, the unique possible utility V_h^E is V_h^*. $\square$

Lemma 22. *A pure strategy equilibrium exists if, and only if,*

$$u'\left[u^{-1}\left(c_l^{FI}\right)\right]\frac{\partial_+\Pi_h\left(V^*\right)}{\partial V_l} < \frac{\pi_l}{\pi_h}.$$

Proof. (If) Consider the following strategy profile. All firms offer mechanism M^{V^*} and consumers follow an optimal strategy that follows truth-telling whenever optimal and treats firms symmetrically. Given the definition of V^*, no firm has a profitable deviation in which a single type is attracted to its offer. This is trivially the case for type l. For type h, such a deviation would require that it offer utility pair V' with $V_h' > V_h^*$ and $V_l' > V_l^*$, which implies $\Pi_h(V') < \Pi_h(V^E) = 0$. Hence, this strategy profile constitutes a pure strategy equilibrium if, and only if, no alternative mechanism can attract both types and generate strictly positive expected profits. But given concavity of $\Pi(\cdot)$, we can find $V' \geq V^*$ such that $\Pi(V') > \Pi(V^*)$ if, and only if, we can find a pair $(d_h, d_l) \geq 0$ such that

$$\frac{\partial\Pi}{\partial V_l}(V^*)d_l + \frac{\partial_+\Pi}{\partial V_h}(V^*)d_h > 0,$$

where we have used the fact that $\Pi(\cdot)$ is differentiable in V_l, whenever $V_l < V_h$. Since $\frac{\partial_+\Pi}{\partial V_h}(V^*) < 0$, such a pair exists if, and only if, $\frac{\partial\Pi}{\partial V_l}(V^*) > 0$.

Condition $\frac{\partial\Pi}{\partial V_l}(V^*) > 0$ coincides with the expression in the lemma since, for V with $V_l < V_h$, we have that $\Pi(V) = \pi_l\Pi^{FI}(V_l) + \pi_h\Pi_h(V_l, V_h)$.

(Only if) If the condition in the Lemma fails, we can find $V' \geq V^*$ such that $\Pi(V') > \Pi(V^*)$. Now consider any pure strategy equilibrium with outcome M^{V^*}. Offer $M^{V'}$ is a profitable deviation by any firm. □