The Combinatorial Multi-Round Ascending Auction*

Bernhard Kasberger† Alexander Teytelboym‡

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Abstract

The Combinatorial Multi-Round Ascending Auction (CMRA) is a new auction format which has already been used in several recent European spectrum auctions. We characterize equilibria that feature auction-specific forms of truthful bidding, demand expansion, and (riskless) demand reduction for settings in which bidders have either decreasing or non-decreasing marginal values. Overall, our results suggest that the CMRA might be an attractive auction design in the presence of highly complementary goods on sale. We discuss to what extent our theory is consistent with outcomes data in Danish spectrum auctions and how our predictions can be tested using bidding data.

Keywords: combinatorial auctions, spectrum auctions, demand reduction, demand expansion, collusion, complementarities.

JEL: D44, D47, L41, L96.

*We are very grateful to Maarten Janssen, Christian Koboldt, Dan Maldoom, Roger Salsas, and Kyle Woodward for their comments and to Edwin Lock for outstanding research assistance. While one could argue that the auction's name ought to be abbreviated to "CMRAA," we follow the designers' original acronym.

†Düsseldorf Institute for Competition Economics (DICE), Heinrich Heine University Düsseldorf, Email: kasberger@dice.hhu.de

‡Department of Economics and St. Catherine's College, University of Oxford. Email: alexander.teytelboym@economics.ox.ac.uk. This work was supported by the Economic and Social Research Council grant number ES/R007470/1.

1 Introduction

Auction sales of radio spectrum have served as a powerful engine for auction format innovation. Spectrum auctions are high-stakes and often involve many heterogeneous spectrum bands requiring flexible and robust auction formats. For example, the Simultaneous Multi-Round Auction (SMRA), the Combinatorial Clock Auction (CCA), and the Deferred Acceptance Auction, all trace their origins to spectrum auction sales. Many dynamic auctions work well—they find an efficient outcome, give bidders decent individual incentives to bid truthfully, and are fairly robust to collusion—when goods for sale are substitutes (Kelso Jr and Crawford, 1982; Milgrom, 2000). In the presence of complements—which is very common in spectrum allocation—auction design becomes very difficult. To help bidders avoid ending up with undesirable combinations of goods, many *combinatorial* auction formats, such as the CCA and the sealed-bid combinatorial auction, allow bidders to express preferences over packages. For example, in the CCA bidders can submit a single package bid in each clock round (that helps discover prices) and any mutually exclusive package bids in the supplementary round (that establishes the winning allocations and payments). In its 2016 spectrum auction, the Danish Energy Agency (DEA) pioneered the *Combinatorial Multi-Round Ascending Auction* (CMRA) format developed by its UK-based consultant DotEcon Ltd. to sell several spectrum blocks in the 1800 MHz band (DotEcon, 2016). Since then, the CMRA has been used twice again by the DEA and once by the Norwegian regulator (Nkom).

The CMRA works as follows. There is a price clock for each good. In each clock round, bidders can report a *headline demand* for a single package which is linearly priced at clock prices. The novel feature of the CMRA is that in each clock round, bidders can also submit *additional* package *bids* at or below the clock prices (and subject to an activity rule). The auction ends when there exists a revenue-maximizing allocation in which exactly one bid from every bidder is accepted.

There are two main differences between the CMRA and the CCA: the CMRA has no supplementary sealed-bid round (while the CCA does) and CMRA allows for non-linearly priced additional bids in the clock stage (while the CCA does not). In the presence of complements, non-linear pricing might be necessary to clear the market. To this end, the CCA typically uses Vickrey–Clarke–Groves (VCG)-like pricing in the supplementary round. By contrast, the CMRA allows bidders to submit non-linearly (pay-as-bid) priced additional bids in the clock phase. Hence, the market can clear in the clock phase, making the supplementary phase unnecessary.[1] A more minor difference is that the CCA ends when there is no excess demand for any good whereas the CMRA ends when the auctioneer can accept a bid from each bidder in a revenue-maximizing allocation.

There are several reasons why one might prefer to use the CMRA in practice. First,

[1] The combinatorial auction due to Porter et al. (2003) also features no supplementary round.

the supplementary round in the CCA creates uncertainty about what the bidders will win and what they will pay. Avoiding the supplementary round can therefore help bidders discover prices more gradually and deal more easily with their budget constraints (Janssen et al., 2017). Second, submitting mutually exclusive bids in the supplementary round (in order to achieve the most preferred auction outcome) can be taxing for bidders. Indeed, bidders might prefer the ability to revise the packages they bid on as prices change. Since the CMRA elicits demand information gradually, one might hope to overcome the "missing bids" problem that has been observed in many CCAs (Bichler et al., 2013).

While a number of theoretical papers have explored the CCA, there has not yet been an analysis of the CMRA. This paper attempts to fill this gap.

In our model, there are two bidders and a single divisible good. We allow for a cap which is a typical feature in spectrum auctions. Bidders' marginal utilities for the good can be decreasing (capturing substitutes) or non-decreasing (capturing (weak) complements). When we look at strategic bidding, we focus on equilibria in proxy strategies (Ausubel and Milgrom, 2002).

We first examine the effects of truthful (non-strategic) bidding. We identify two variants of truthful bidding in the CMRA. First, the headline demands can be truthful while not submitting any additional bids. We call this strategy *clock-truthful* as it mimics truthful bidding in a clock auction. The second variant of truthful bidding features truthful headline demands and additional bids. Specifically, under this so-called *CMRA-truthful* strategy, bidders submit additional bids that give them the same surplus as the headline demand (so bidders are indifferent between winning with their headline demand or any additional bid). As bids can only be non-negative, bidders do not immediately bid on all packages; the range of packages that receives non-negative bids increases as the clock price goes up. The CMRA-truthful strategy was mentioned by DotEcon (2016).

The first insight is that truthful additional bids end the auction at a lower clock price than had there been no additional bids as under clock-truthful bidding. Under CMRA-truthful bidding, bidders express true marginal values on a larger range of quantities at lower clock prices. It is then natural that the auction ends at a lower clock price and with lower revenue. Indeed, we show that both properties are true irrespective of bidders having increasing or decreasing marginal values (Theorems 1 and 2).

The second insight is that truthful additional bids can restore efficiency. Consider clock-truthful bidding when bidders have increasing marginal values. Bidders demand the maximum capped quantity until they are indifferent between winning it and winning nothing; they drop demand to zero. Hence, during the auction they did not express any marginal values between zero and the maximum allowed quantity; the auction ends with excess supply. However, under CMRA-truthful bidding bidders use the additional bids to report their marginal values for quantities they never bid on in the clock auction, leading to the efficient allocation. Theorems 1 and 2 show that CMRA-truthful bidding leads to the efficient allocation irrespective of increasing or decreasing marginal values.

Next, we turn to the strategic incentives. Clock-truthful bidding is never an equilibrium: with decreasing marginal values, bidders have demand reduction incentives as in other multi-unit auctions (Ausubel et al., 2014) and with increasing marginal values the weaker bidder has an incentive to close the auction early by bidding on the units left over by the stronger bidder (who bids up to her cap). CMRA-truthful bidding is also not an equilibrium with decreasing marginal values due to a free-riding and threshold problem (Theorem 3).

Remarkably, CMRA-truthful bidding turns out to be an ex-post equilibrium when bidders have increasing marginal values (Theorem 4). We highlight two features of the auction design that make CMRA-truthful bidding an equilibrium. For both features it is important to bear in mind that the truthful headline demand is the highest possible (capped) quantity under increasing marginal values (as long as any quantity is demanded at the clock price). First, bidders have the flexibility to place low-valued additional bids on small quantities at high clock prices while keeping their headline demand at a large quantity. This flexibility makes the deviation of weak bidders as in the clock auction unnecessary. Recall that in the clock auction weak bidders have an incentive to demand a small quantity at a low clock price when the other bidder is expected to only demand large quantities. In the CMRA (weak) bidders can make the same bids also a high clock prices through the additional bids. The second feature is the closing rule. Unlike the clock auction, the CMRA does not end as soon as there is no excess demand (in headline demands). Instead, the CMRA ends when the revenue-maximizing allocation contains one bid by each bidder. A consequence is that any deviating bid must be, intuitively, sufficiently high to outbid the other bidders' (large) headline demand. The pay-as-bid pricing rule makes such high deviating bids unattractive.

We also characterize equilibria when bidders have decreasing marginal values. In particular, there is an inefficient ex-post equilibrium in demand expansion strategies in which bidders submit a constant headline demand for the maximum (capped) quantity and one additional bid of zero (i.e., the reserve price) for the residual supply at a relatively high clock price. In this equilibrium, the stronger bidder wins a larger quantity than what he would have won in the efficient allocation. The same *constant* strategy also forms an ex-post equilibrium under increasing marginal values. In fact, the constant strategy turns out to be outcome-equivalent to CMRA-truthful bidding under increasing marginal values (Theorem 5) and to a VCG auction with truthful bidding.

Finally, we show how bidders can use the additional bids to collude. Suppose at the start of the clock, each bidder places a headline demand of the maximum capped quantity and an additional bid of half the initial clock price for half of the available quantity. As both bidders winning half of the supply is (weakly) revenue-maximizing, the auction ends immediately by assigning half of the supply to each bidder at the initial clock

Marg. values	Bidding strategy Clock-truthful		CMRA-truthful		Constant		RDR	
Decreasing	Efficient	(1)	Efficient	(2)	Inefficient	(5)	Inefficient	(6)
	Not eqm		Not eqm	(3)	Ex-post eqm	(5)	BNE (more likely)	(6)
Non-decreasing	Inefficient	(2)	Efficient	(2)	Efficient	(5)	Inefficient	(6)
	Not eqm		Ex-post eqm	(4)	Ex-post eqm	(5)	BNE (less likely)	(6)

Figure 1: Results for different strategies in the CMRA.
Notes: In parentheses, we indicate the Theorem that proves the result in our model. RDR = riskless demand reduction; eqm = equilibrium; BNE = Bayes-Nash equilibrium.

price. We show that this strategy profile forms an equilibrium if each bidder expects the opponent to be sufficiently strong. Moreover, if the bidders' marginal values for each unit decrease, then the equilibrium conditions for collusion become easier to satisfy (Theorem 6). Importantly, this collusive demand reduction strategy carries no risk. If the other bidder does not work out the collusive bidding strategy and the auction does not end in the first round, the bidder is unconstrained in her bids in later rounds.

Taken together, our results suggest that the CMRA is a surprisingly effective auction format for selling complements: not only is a salient version of straightforward bidding an efficient, ex-post equilibrium, but collusion is also less likely to be profitable. By contrast, the CMRA does not have appealing properties for selling substitutes where a clock auction could raise more revenue. Fig. 1 summarizes our results.

We then connect our theoretical predictions to outcomes of CMRAs in practice. We look at outcomes of three Danish spectrum auctions held between 2016 and 2021. As the DEA only publishes data on allocations and total bidder payments, we try to reverse-engineer whether headline demands or additional bids were winning in the auctions. We then use these insights to speculate whether the possible bidding dynamics were consistent with our theoretical predictions. It appears that in the 2016 auction only headline demands won, which suggests that bidders behaved as if they were in a clock auction. In the 2019 and 2021 auctions, some additional bids were winning, suggesting that bidders were using much richer strategies in later auctions. However, we do not find any evidence that bidders use the riskless collusion strategy that we outline. Finally, we explain what our results mean for patterns of bidding dynamics and how our theoretical predictions could be tested on actual CMRA bidding data.

The literature on combinatorial auctions is vast. Combinatorial auctions were introduced in the context of selling airport take-off and landing slots by Rassenti et al. (1982). Following the success of the SMRA for early US spectrum auctions in 1990s (Milgrom, 2000), license complementarities in subsequent sales were substantial enough to warrant new auction formats. As a result, a number of combinatorial auction designs were proposed (Parkes and Ungar, 2002; Porter et al., 2003).[2]

[2] Palacios-Huerta et al. (2021) offer an excellent recent overview of the use of combinatorial auctions, including CMRAs, in practice.

A major breakthrough was the development of the CCA (Ausubel et al., 2006; Maldoom, 2007) which quickly became one of the dominant auction formats for spectrum sales and beyond. The CCA was theoretically analyzed by Levin and Skrzypacz (2016), Janssen and Karamychev (2016) and Janssen and Kasberger (2019). We use many modelling features from the elegant analysis of Levin and Skrzypacz (2016) as well as their focus on proxy strategies. Unlike many other papers, we consider both increasing and decreasing marginal values.

Many papers have studied demand reduction in multi-unit auctions (e.g., Brusco and Lopomo (2002); Grimm et al. (2003); Ausubel et al. (2014)). The main difference is that in our case the demand reduction strategy is riskless while in other auction formats it is not. We also study demand expansion which has not been noted in auctions with pay-as-bid pricing.

Sun and Yang (2014) and Baranov et al. (2017) proposed innovative auction designs for selling pure complements. Sun and Yang's dynamic auction for heterogeneous indivisible complementary goods converges to an efficient allocation supported by a non-linear-pricing Walrasian equilibrium (i.e., there is a market-clearing price for each package rather than for each good). In their auction, truthful bidding is an ex-post equilibrium and bidders' payments coincide with VCG payments. In contrast, we derive results for the CMRA in settings where units of a homogeneous good can be either complementary or substitutable and point out the possibility of collusive bidding.

Baranov et al. (2017) considered bidders with increasing marginal values and augment the clock auction to let bidders express the interval of quantities that are acceptable at the current clock price. As no (additional) bids are made, the auctioneer needs to keep track of the implicitly expressed values. They show that truthful bidding is an ex-post equilibrium in their auction and that the bidders' payments equal VCG prices. In contrast, the CMRA features a richer bidding language as bids on any quantities can be made (and not just intervals) and bidders can express values for these quantities. Hence, there is no need for the auctioneer to infer anything from the values.

Finally, Baranov (2018) demonstrated that eliciting marginal values in addition to making demand queries can help find efficient outcomes in a monotonic clock auction.[3]

We proceed as follows. The next section illustrates the dynamics of the clock auction and the CMRA under different bidding strategies. Section 3 presents the formal model and the details of the rules of the CMRA. In Section 4, we analyze non-strategic truthful bidding and then we turn to strategic bidding in Section 5. We describe how CMRA performed in the Danish spectrum auctions in Section 6 where we also explain how to use bidding data to test which equilibrium bidders are likely to be playing. Section 7 is a conclusion. All omitted proofs and further details of Danish auction are in the Appendix.

[3]Sun and Yang (2009) and Teytelboym (2014) proposed dynamic auctions for selling restricted classes of substitutes and complements.

2 Illustrative Example: Clock Auction vs. CMRA

To fix ideas, we illustrate the differences between the clock auction and the CMRA with a simple example. There are four identical lots for sale and two bidders. Bidders are symmetric: The valuation of each lot is \$30 and the valuations are additive for each bidder. There is a symmetric cap of three lots for either bidder.

In a (standard) clock auction the price starts at \$0 and increases as long as there is excess demand. Under truthful bidding, each bidder demands three lots as long as the clock price is below \$30. At \$30 the clock stops because both bidders drop their demands to 0. The bidders have expressed demand for three units which can be allocated to either bidder, therefore there is excess supply of one lot. Hence, the final allocation is inefficient. The auction revenue is \$90.[4]

Next, consider how the CMRA would run. Bidders can mimic their strategies in the clock auction by only submitting a headline demand (and no additional bids) at each clock price. If the headline demands are truthful—this is the *clock-truthful* setting—and the price is below \$30, then the revenue-maximizing allocation involves only *one* bidder therefore the price continues to increase. At \$30, bidders drop their headline demand to zero lots. Now the revenue-maximizing allocation can involve a bid from both bidders: we can allocate three units to one bidder and no units to the other bidder, therefore the outcome is the same as in the clock auction. It is easy to see why clock-truthful bidding does not constitute an equilibrium: Bidders do not get any surplus from winning three lots, so a better strategy is to submit a headline demand of one unit in the initial round. This would end the auction immediately with both bidders having a positive payoff.

Now consider the presence of additional bids in the CMRA (headline demands continue to be three units at prices below \$30). Fig. 2 illustrates the *CMRA-truthful* strategy in our example. At any clock price up to \$10, the bidder only submits a headline demand of three units. At the clock price of \$10 the bidder is indifferent between winning three lots at linear prices (with a surplus $3 \times \$20 = \$60$) and winning two lots at a price of zero. Hence, the bidder submits an additional bid for a total of \$0 for two lots (note that the price in this additional bid is below the clock price). Indeed, if both bidders submit such bids, there exists an allocation of all four lots to both bidders such that exactly one bid is accepted from each bidder, but it is not revenue maximizing (i.e., it raises \$0 compared to the allocation of three units to one bidder which raises \$30). Hence the auction continues. As the clock price increases, the bidder changes the price she submits for the additional bid of two lots. For example, at a clock price of \$12, the bidder submits an additional bid of \$6 for two lots. Then, at a clock price of \$20, the bidders would continue with their additional bid of \$30 for two lots and add an additional bid of \$0 for one lot because

[4]Depending on the design of the clock auction, it is possible that when bidders drop their demand to zero simultaneously, the auctioneer allocates nothing to the bidders. In this case, imagine that the valuation of one of the bidders is $\$30+\epsilon$.

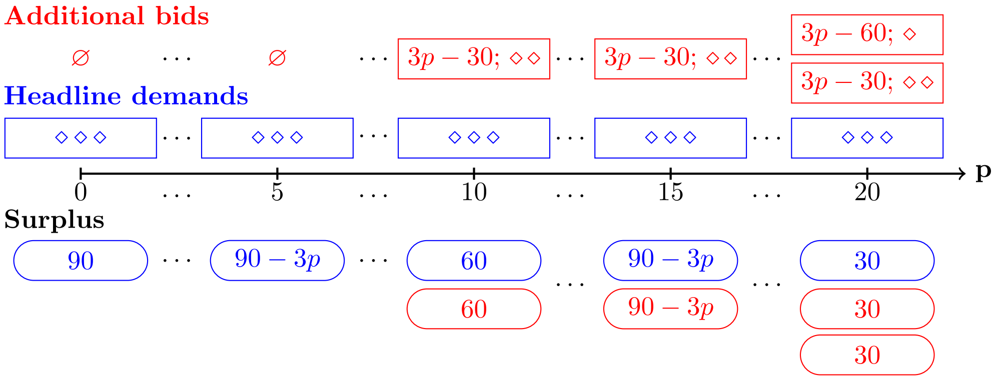


Figure 2: Headline demands and additional bids under CMRA-truthful bidding strategy in the illustrative example.

Notes: Above the price axis: Diamonds indicate the number of units in the bid; headline demands are at the clock price p and additional bids are at stated prices. Below the price axis: Surplus from each individual bid.

the surplus from each bid is \$30. Now there are revenue-maximizing allocations in which an auctioneer can allocate (all four) units and in which exactly one bid of each bidder is accepted: either the bidders' additional bids on two lots are accepted or one bidder's headline demand of three lots and one bidder's additional bid for one lot are accepted. In this setting, the final allocation is efficient, but revenue is lower (\$60 vs. \$90) than in the clock auction. We show that, remarkably, CMRA-truthful strategies constitute an ex-post equilibrium in proxy strategies whenever bidders' marginal values are constant (as in this example) or increasing.

However, there turns out to be another equilibrium in which bidders can risklessly collude. In the first round of the auction, the bidders follow the CMRA-truthful strategy, but also add an additional bid for two lots at a price of \$0. In our example, the auction ends immediately and both bidders win two units each at a price of zero. If the auction does not end in the first round, the bidders submit CMRA-truthful bids in all subsequent rounds and the auction will proceed as in case of CMRA-truthful equilibrium.[5] Note that this first-round demand reduction carries no risk for the bidders. Specifically, bidders do not have to give up "eligibility," that is, they do not have to lower their headline demand to implement a collusive outcome. The clock auction, on the other hand, allows demand reduction only in headline demands; hence, lowering demand to two lots in early rounds does not allow the bidder to win more units later.

The following model is different from the illustrative example in two ways. First, we assume for tractability that the good is divisible. Second, we do not assume that marginal values are constant; instead we also allow for decreasing and increasing marginal values.

[5] In fact in this example, even clock-truthful bids and no additional bids would suffice for the collusive equilibrium.

3 The Combinatorial Multi-Round Ascending Auction

The CMRA has been used for selling multiple units of different goods to many potential buyers. For clarity, we focus on a setting with two bidders (denoted by $i \in \{1,2\}$) and one unit of a perfectly divisible good whose quantity is denoted by $x \in [0,1]$. We therefore map the auction rules of the 2016 Danish spectrum auction for the 1800 MHz band to our setting.

Clock and headline demands. The core of the CMRA is a clock auction: There is a price clock $p \in \mathbb{R}_+$ that ticks upwards, and bidders report their demands at each clock price p. The demanded quantity is called the *headline demand*. Formally, at clock price p bidder i has a headline demand $h_i(p) \in [0,1]$.

Additional bids. The innovation of the CMRA is that bidders can submit *additional bids* in each clock round. Formally, the additional bid for quantity x when the clock price is p is the price (willingness-to-pay) $A_i(x;p) \in \mathbb{R}_+ \cup \{-\infty\}$. In contrast to the headline demands, the additional bids do not have to be linear in clock prices. Instead, they must be below clock prices, i.e., $A_i(x;p) \leq px$. Moreover, additional bids must either be zero or $-\infty$ for the empty package and must satisfy the activity rule described below. For quantities that do not receive any additional bids we define the additional bids to be negative, i.e., $A_i(x;p) = -\infty$ if bidder i does not submit an additional bid on quantity x.

At each clock price, the headline demand and the additional bids create a bid function $B_i \colon [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ that maps quantities and the clock price to bids. The bid function $B_i(\cdot;p)$ is then the collection of highest bids expressed in the course of the auction by bidder i at clock price p. For the headline demand the bid function simply gives us $B_i(h_i(p);p) = p \cdot h_i(p)$ while for the additional bids the bid function is defined as $B_i(x;p) = \sup_{\tilde{p} \leq p} A_i(x;\tilde{p})$.

Closing rule. The auction ends if there is a feasible revenue-maximizing allocation in which a bid by every bidder is accepted. Formally, the auction ends at the clock price p if there is a non-negative allocation $(x_1, x_2) \in [0,1]^2$, where (x_1, x_2) is such that

1. $(x_1, x_2) \in \arg\max_{(\tilde{x}_1, \tilde{x}_2) \in [0,1]^2,\, \tilde{x}_1 + \tilde{x}_2 \leq 1} B_1(\tilde{x}_1;p) + B_2(\tilde{x}_2;p)$, and

2. $B_i(x_i;p) \geq 0$ for $i \in \{1,2\}$.

The second requirement guarantees that bidders win nothing only if they bid on zero quantity because the bid on the empty package must be zero by the auction rules above. If there is no such allocation, the auction continues by raising the clock price p.

It is worth comparing the closing rule to those of the clock auction and of the CCA. The clock auction ends as soon as there is no excess demand. A bidder wins nothing only if she drops her demand to zero. The final allocation need not maximize revenue as it does not take past bids into account (unlike the CMRA). The clock phase in the CCA ends when there is no excess demand. Depending on the specific activity rules and the final clock round, it is possible that bidders do not win anything. The CCA selects the final allocation as in closing rule condition (1) above (i.e., ignoring condition (2)), but the bidders' payments are weakly lower than B_i.

Activity rule. The headline demand determines the bidders' eligibility for the quantities they can bid on. Let $p < p'$ and $h_i(p) > h_i(p')$. Suppose p was the highest price at which $h_i(p)$ was demanded and that for all prices $\tilde{p} \in (p, p')$ bidder i demanded $h_i(p')$. For quantities $x \in (h_i(p'), h_i(p))$ we have a "relative cap" on $B_i(x; \tilde{p})$ that constrains $B_i(x; \tilde{p})$ from above by[6]

$$B_i(x; \tilde{p}) \leq B_i(h_i(p')) + p'(x - h_i(p')).$$

Strategies. A *proxy strategy* $(h_i, (A_i(\cdot; p))_{p \geq 0})$ for bidder i consists of a headline demand h_i and a collection $(A_i(\cdot; p))_{p \geq 0}$ of additional bids indexed by clock prices p. Clearly, a proxy strategy must satisfy the activity rule. Similar to Levin and Skrzypacz (2016), we study equilibria in proxy strategies to focus on outcomes that are specific to the CMRA. Levin and Skrzypacz (2016) point out that dynamic auctions typically feature other equilibria in which certain actions trigger reactions of the other bidders. The restriction to proxy strategies avoids the specification of (updated) beliefs about the other bidder's type, which does not play a major role in our analysis due to our focus on ex post equilibria.[7]

Information. In each round, bidders learn (i) the current clock prices, (ii) whether their previous headline demand appeared in a revenue-maximizing allocation, (iii) whether they appeared in a revenue-maximizing allocation. Due to our focus on proxy strategies, (ii) and (iii) will play no role to our analysis.[8]

Payoffs. Bidder i has a quasilinear utility function $U_i(x) - px$, where $U_i(x)$ is the value of winning quantity x. Let $u_i(x)$ denote the marginal value of the x^{th} unit so

[6] If headline demand is x and lowered continuously at clock price p, the slope of the bids at x can be at most p.

[7] Moreover, proxy strategies only specify "on-path" behavior; as the proxy strategy rules out certain bidding histories, the strategy does not specify behavior after such histories. Specifically, there are sequences of additional bids that put non-trivial constraints on the bidding rules. Specifying behavior in such cases is notationally complex. We avoid these issues by considering proxy strategies.

[8] For example, in the 2021 Danish spectrum auction bidders learned (i)–(iii), while in the 2016 and 2019 auctions they only learned (i).

$U_i(x) = \int_0^x u_i(y)dy$. Marginal values are strictly positive. There is a *cap* $\lambda \in (1/2, 1)$ such that bidder i can win at most λ.[9] We assume that both bidders face the same cap for simplicity. Spectrum caps were present in all real-world implementations of the CMRA.

The value functions are twice continuously differentiable and parameterized by private types θ_i, i.e., $U_i(x) = U(x; \theta_i)$, where $\theta_i \in [\underline{\theta}, \overline{\theta}]$ and $\underline{\theta}, \overline{\theta} \in \mathbb{R}_+$. We say that bidder i is *stronger* than bidder j if $\theta_i \geq \theta_j$. Let values be strictly increasing in θ_i, i.e., $\partial U(x; \theta_i)/\partial \theta_i > 0$ for all $x \in [0, \lambda]$. Let the marginal values be strictly increasing in θ_i, that is, $\partial U/\partial x \partial \theta_i > 0$. We assume that types are independently and identically distributed. The cumulative distribution function F is continuously differentiable and has density f.

Our analysis distinguishes strictly decreasing marginal values (capturing substitutes) and non-decreasing marginal values (capturing independent goods and complements). If bidders have strictly decreasing marginal values on $[0, \lambda]$, then we also assume that the efficient allocation is interior, i.e., $0 < x_i^\star < \lambda$. The following assumption contains a sufficient condition.

Assumption 1. Bidders have *strictly decreasing marginal values* if $u_i'(x) < 0$ for $x \in [0, \lambda]$ and $u_i(\lambda) < u_j(1 - \lambda)$ for $i, j = 1, 2$

When bidders have strictly decreasing marginal values, the efficient allocation $x^\star$ solves $\max U_1(x_1) + U_2(x_2)$ subject to $x_1 + x_2 \leq 1$. The assumptions imply that $x_1^\star = 1 - x_2^\star$ and that $u_1(x_1^\star) = u_2(1 - x_1^\star)$.

We also analyze settings with non-decreasing marginal values.

Assumption 2. Bidders have *non-decreasing marginal values* if $u_i'(x) \geq 0$ for $x \in [0, \lambda]$ for $i = 1, 2$.

Under non-decreasing marginal values, the efficient allocation is such that the bidder with the higher willingness-to-pay as measured by the type θ wins λ. Formally, the efficient allocation $x^\star$ is $x_i^\star = \lambda$ and $x_j^\star = 1 - \lambda$ if $\theta_i \geq \theta_j$. To see this, first note that the efficient allocation must lie on the boundary due to non-decreasing marginal values. Now observe that $U_i(\lambda) + U_j(1 - \lambda) > U_i(1 - \lambda) + U_j(\lambda)$ is equivalent to $\int_{1-\lambda}^{\lambda} u_i(x)dx > \int_{1-\lambda}^{\lambda} u_j(x)dx$. As the marginal values are increasing in θ, the inequality holds. Finally, note that non-decreasing marginal values includes constant marginal values.

4 Outcomes under non-strategic truthful bidding

In this section, we investigate outcomes under non-strategic truthful bidding. Our motivation is three-fold. First, truthful strategies allow us to introduce the workings of the

[9] An alternative interpretation is that λ is bidder i's capacity (Ausubel et al., 2014) or satiation point where marginal value at or below the reserve price. In this case, we would define $U_i(x)$ to be strictly increasing on $[0, \lambda]$ and flat on $(\lambda, 1]$, that is, $U_i(x) = U_i(\lambda)$ for $x \in [\lambda, 1]$.

CMRA without the need to keep track of incentives. Second, truthful strategies are a reasonable approximation of behavior in auctions with many bidders, so they provide a natural benchmark. Lastly, these strategies set the stage for the strategic incentives studied in the next section.

We compare the simple clock auction to the CMRA. As the CMRA is built on a clock auction, any clock auction behavior is also feasible in the CMRA; bidders only have to ignore the possibility of additional bids.

In the clock auction, the meaning of truthful bidding is unambiguous. Bidders can demand only a single quantity at a time, and this quantity is utility-maximizing if it is truthful. As there is a unique headline demand available in each round, we take the notion of truthful clock-auction bids for defining truthful headline demand. Thus, headline demand $h(p)$ in the CMRA is *truthful* at current price p if $h_i(p) \in \arg\max_x U_i(x) - px$. We say that a bidding strategy in the CMRA is *clock-truthful* if the bidder only submits truthful headline demands at all clock prices (and no additional bids).

The CMRA's additional bids allow richer behavior that may be considered truthful. Truthful headline demand at all prices elicits a demand curve that reflects the bidder's marginal values. This motivates us to interpret truthful bidding in the CMRA as expressing true marginal values for all shares for which this is possible at a given clock price. The following "truthful" bidding strategy was described by DotEcon:

> *The proposed auction design allows bidders to follow alternative bid strategies that give them more control over their risks. For instance, a bidder may submit many bids at all times with a view to maximise surplus. Making a headline bid for the surplus-maximising package and then for all other packages with a bid amount equal to the value of the package minus the surplus on the headline bid would achieve this. (DotEcon, 2016)*

In other words, DotEcon suggests that bidders place an additional bid for x such that the bidder is indifferent between winning x for a payment of $A(x;p)$ and the headline demand at price p. Intuitively, the bidder would then bid true marginal values for all shares for which this is possible with non-negative bids.

Formally, we say that a bidding strategy is *CMRA-truthful* if the headline demands are truthful and

$$A(x;p) = \begin{cases} U(x) - V(p) & \text{for } x \in [U^{-1}(V(p)), \lambda] \\ -\infty & \text{otherwise,} \end{cases} \tag{1}$$

where $V(p) = \max_x U(x) - px$. The additional bids are generically non-linear in clock price p.

The CMRA-truthful bids have the following properties. For $p = 0$, the truthful demand is λ so that $V(0) = U(\lambda)$ and $U^{-1}(U(\lambda)) = \lambda$. So only λ receives a non-negative bid at $p = 0$. The envelope theorem implies that $V(p)$ decreases in p, implying that the range of quantities that receive non-negative bids increases in p. Moreover, the bid on a given quantity increases in the clock price. Note that as p tends to infinity, $V(p)$ converges to 0. Thus, for sufficiently high prices the bidder bids on all quantities in $[0, \lambda]$ and the additional bids are the true valuation levels. Hence, the domain of the non-negative additional bids is well defined. Bidders shade their bids by bidding less than value without distorting marginal values for quantities for which a positive bid is made.

4.1 Decreasing marginal values

We first analyze auction outcomes when bidders have decreasing marginal values.

Theorem 1. *Suppose that marginal values are strictly decreasing (Assumption 1). Then:*

1. *Clock-truthful bidding in the CMRA leads to the efficient allocation. The clock ends at* $p^\star = u_1(x_1^\star) = u_2(x_2^\star)$.

2. *CMRA-truthful bidding leads to the efficient allocation and ends the auction at a price lower than* $p^\star$.

3. *Ex-post revenue under CMRA-truthful bidding is lower than under clock-truthful bidding.*

With decreasing marginal values, truthful bidding in the clock auction (Levin and Skrzypacz, 2016) and clock-truthful bidding in the CMRA both lead to the efficient allocation. The revenue is $p^\star$. CMRA-truthful bidding also leads to the efficient allocation but ends the auction at a lower price and at lower revenue. The reason why the CMRA ends earlier than the clock auction with the efficient allocation is that bidders can express true marginal values for more quantities at lower clock prices.

Fig. 3 illustrates Theorem 1. Fig. 3a shows the headline demands and additional bids at various clock prices. Bidder 1 is stronger and her efficient share is $x_1^\star > \frac{1}{2}$ while Bidder 2's efficient share is $x_2^\star < \frac{1}{2}$. Solid lines are headline demands $h_i(p)$ and dashed lines are additional bids $A_i(x; p)$ as in Eq. (1). Fig. 3b depicts the respective revenue from feasible allocations. The solid line $B_1(x; p) + B_2(1 - x; p)$ shows revenue for allocations in which a bid of each bidder is accepted since this is required by the CMRA closing rule (recall that bids are $-\infty$ for shares that bidders do not bid on). The dashed line is $\max\{B_1(x; p), B_2(1 - x; p)\}$ for allocations that do not receive non-negative bids from both bidders: this is the revenue that would be obtained by accepting one bidder's bid.

Let us consider how the bids and allocations change as the clock price increases. As a benchmark, consider a simple clock auction (or a CMRA with clock-truthful bidding). We simply increase the clock price and follow the headline demands in solid lines in Fig. 3a. The auction ends at clock price $p^\star$ with market clearing.

In the CMRA under CMRA-truthful bidding, both bidders submit headline demands and additional bids. When the clock price p is low, only quantities close to λ receive additional bids.

- *Clock price* p_1. At this clock price, each bidder's headline demand is λ and each bidder's surplus is $U_i(\lambda) - \lambda p_1$. Recall that the additional bids are given by Eq. (1). At p_1, the additional bids range from λp_1 (for a quantity λ) to zero (for a smaller quantity that keeps the bidder indifferent). At p_1, there is no feasible allocation that receives bids from both bidders: in Fig. 3b, the bidders' additional bids do not intersect. The auction continues as it is not possible to accept a bid by each bidder in the revenue-maximizing allocation.

- *Clock price* p_2. This is the lowest price at which both bidders bid on their respective efficient quantities. Bidder 1 submits a strictly positive additional bid on $x_1^\star$ at clock price p_2, while Bidder 2 submits an additional bid of 0 on $x_2^\star$. From now on the efficient allocation can in principle be allocated as it receives bids from both bidders. Fig. 3b reveals, however, that the efficient allocation is not revenue-maximizing. Bidder 1's headline demand is still λ, and allocating λ to Bidder 1 raises λp_2 revenue. Observe that Bidder 2 bids less than λp_2 on λ because marginal values are decreasing and because λ is not the headline demand. As Bidder 2 does not bid on $1 - \lambda$ at clock price p_2, the revenue-maximizing allocation features only bids by one bidder and the auction continues.

- *Clock price* p_3. Both bidders have now raised their additional bids on their respective efficient share. As we can see in Fig. 3b, the efficient allocation $x^\star$ locally maximizes revenue. However, $x^\star$ does not yield a global revenue maximum as Bidder 1's bid on λ leads to a higher revenue of λp_3.

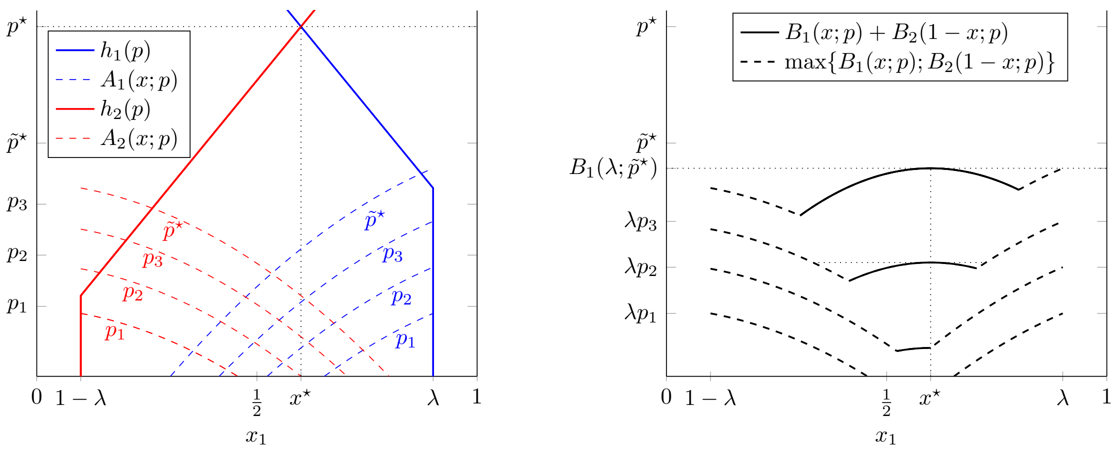
- *Clock price* $\tilde{p}^\star$. Now, the additional bids are sufficiently high so that the efficient allocation $x^\star$ is revenue-maximizing. The auction ends at $\tilde{p}^\star$ at which

$$B_1(x_1^\star; \tilde{p}^\star) + B_2(x_2^\star; \tilde{p}^\star) = \max_i B_i(\lambda; \tilde{p}^\star), \tag{2}$$

  where Bidder 1 has the higher bid on λ in our example. Note that at price $\tilde{p}^\star$ Bidder 2 does not yet bid on $1 - \lambda$, so $(\lambda, 1 - \lambda)$ is not a feasible allocation. The CMRA ends at clock price $\tilde{p}^\star$. Revenue is clearly lower than $\tilde{p}^\star$ and lower than $p^\star$.

4.2 Non-decreasing marginal values

We now consider a setting with non-decreasing marginal values, which has received a lot less attention in the literature than the setting with decreasing marginal values.



(a) Bidding functions

(b) Revenue of various allocations

Figure 3: CMRA-truthful bidding with decreasing marginal values.
Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

Theorem 2. *Suppose that marginal values are non-decreasing (Assumption 2). Then:*

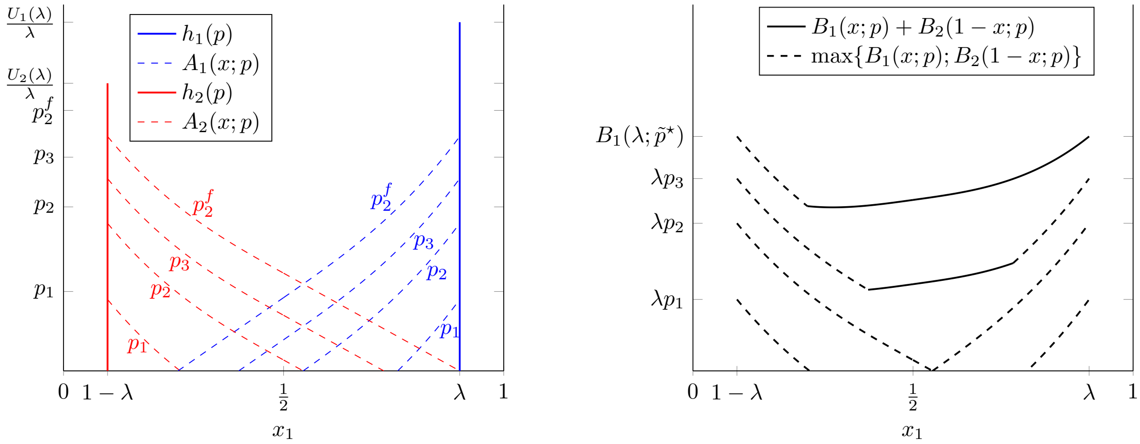
1. *Clock-truthful bidding in the CMRA leads to excess supply. The clock ends at* $p = \min_i U_i(\lambda)/\lambda$. *The final auction allocation is inefficient.*
2. *CMRA-truthful bidding clears the market and leads to the efficient allocation.*
3. *Ex-post revenue under CMRA-truthful bidding is lower than under clock-truthful bidding.*

Theorem 2 already reveals the power of additional bids in the CMRA. The presence of additional bids ensures market-clearing and an efficient allocation while their absence leads to inefficiency and excess supply.

Fig. 4 illustrates the theorem. As before, Fig. 4a shows the headline demands and the additional bids, while Fig. 4b shows the revenue under different allocations.

Once again, let us first consider the outcome of a clock auction or of clock-truthful bidding in the CMRA. Fig. 4a shows that, due to increasing marginal values, bidder i's clock-truthful headline demand is λ for $p \leq U_i(\lambda)/\lambda$ and 0 for higher prices. Hence, the auction ends at clock price $p = \min_i U_i(\lambda)/\lambda = U_2(\lambda)/\lambda$. As Bidder 2 drops demand to 0 at price $U_2(\lambda)/\lambda$, the auction ends with excess supply of $1-\lambda$. Bidder 1 wins quantity λ and the revenue is $U_2(\lambda)$.

In the CMRA under CMRA-truthful bidding, bidders' headline demands are as under clock-truthful bidding but they also submit additional bids. For low clock prices, bidders submit few additional bids, but as the clock price rises, bidders increase their additional bids both on the intensive and extensive margins.



(a) Bidding functions

(b) Revenue of various allocations

Figure 4: CMRA-truthful bidding with non-decreasing marginal values

Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

- *Clock price* p_1. There is no feasible allocation that receives non-negative bids from both bidders. Hence, the auction continues.

- *Clock price* p_2. There are now feasible allocations that receive non-negative bids from both bidders. These allocations are not revenue-maximizing as $B_i(\lambda;p) = \lambda p$ yields higher revenue.

- *Clock price* p_3. Feasible allocations that receive non-negative bids from both bidders are still not revenue-maximizing. Bidder 1's marginal values (and additional bids) are higher and non-decreasing, so allocating more to Bidder 1 increases revenue.

- *Clock price* p_2^f. At this price, the weaker Bidder 2 places an additional bid of 0 on $1-\lambda$. More generally, there is a *final price* p_i^f at which bidder i bids 0 on $1-\lambda$ as this bidder is indifferent between winning λ for a payment λp_i^f and winning $1-\lambda$ for free. The indifference condition

$$U_i(\lambda) - p_i^f \lambda = U_i(1-\lambda)$$

transforms to

$$p_i^f = \frac{U_i(\lambda) - U_i(1-\lambda)}{\lambda}.$$

With Bidder 2's additional bid, it is now possible to accept a bid by each bidder in the revenue-maximizing allocation $(\lambda, 1-\lambda)$ (Fig. 4b). Therefore, the CMRA ends in market-clearing. The revenue is λp_2^f, which is lower than $\lambda U_2(\lambda)/\lambda$.

5 Outcomes under strategic bidding

Having characterized the outcomes under non-strategic truthful bidding, we turn to the strategic incentives in the CMRA. Our starting point are the truthful strategies, for which we show when they constitute equilibrium strategies and when they do not. We then characterize an ex-post equilibrium in proxy strategies and illustrate how the CMRA can be gamed.

Throughout the section we use two solution concepts. The first is Bayes-Nash equilibrium in proxy strategies. Bidders choose proxy strategies that are best responses to their opponent's proxy strategy and the type distribution F. Hence, the proxy strategy is optimal in expectation given variation in the opponent's type. The second is ex-post equilibrium in proxy strategies. Bidders choose proxy strategies that are best responses to the opponent's strategy and any type distribution. In particular, bidders would not choose an alternative proxy strategy if they knew the opponent's type. Note that any ex-post equilibrium is a Bayes-Nash equilibrium. Hence, if a strategy does not constitute a Bayes-Nash equilibrium, it also does not form an ex-post equilibrium.

5.1 Truthful outcomes under strategic bidding

We first examine whether the clock-truthful or CMRA-truthful strategies form equilibria when bidders have decreasing or non-decreasing marginal values.

5.1.1 Decreasing marginal values

Recall the CMRA pricing rule: winning headline demands are priced at the clock price and winning additional bids are priced as bid. Such pricing features appear in clock auctions for multiple units and (sealed-bid) discriminatory multi-unit auctions. A common property of equilibria in these auctions is that bidders have an incentive to "shade" their bids below their value (Ausubel et al., 2014). This intuition turns out to carry over to the CMRA when bidders have decreasing marginal values.

Remark 1. Clock-truthful bidding is not an equilibrium in the CMRA when bidders have decreasing marginal values just like truthful bidding is not an equilibrium in the clock auction. As bidders pay px for winning x at final clock price p, they have an incentive to report a marginally lower willingness-to-pay to end the auction at a lower price as this saves them money on units they win. Such (marginal) deviations are possible only if the other bidder uses a downward sloping demand curve as is natural with decreasing marginal values.

The next theorem states that CMRA-truthful bidding also does not form an equilibrium in the CMRA when bidders have decreasing marginal values.

Theorem 3. *Suppose that marginal values are strictly decreasing (Assumption 1). Then there is no Bayes-Nash equilibrium in proxy strategies in the CMRA in which bidder i bids CMRA-truthfully.*

The proof of the theorem can be illustrated by what happens at clock price p_3 in the example in Section 4.1 where bids are such that the efficient allocation yields a local, but not global, revenue maximum. Bidders face a threshold problem[10] and face two kinds of deviation incentives captured by Eq. (2). First, bidders have an incentive to free-ride on their competitor by shading their bids on (what the bidders expect to be the) efficient shares even more, i.e., bidding lower than dictated by the truthful additional bids. In particular, bidders have a greater incentive to decrease the additional bids more on lower quantities in order to reduce their payments without affecting the allocation significantly. That is, in Eq. (2), Bidder 1 prefers to keep her bid $B_1(x_1^\star; p)$ low and for Bidder 2 to increase $B_2(x_2^\star; p)$. Second, bidders also have an incentive to shade their bids on high quantities. This allows stronger bidders to remove the threshold problem and close the auction early. In Eq. (2), a bidder might wish to decrease $B_i(\lambda; \tilde{p}^\star)$. As a result, CMRA-truthful bidding is not an equilibrium.

5.1.2 Non-decreasing marginal values

Next, we turn to non-decreasing marginal values. Unsurprisingly, clock-truthful bidding is also not an equilibrium when bidders have non-decreasing marginal values.

Remark 2. Under clock-truthful bidding, bidders demand λ and the auction ends at the clock price at which the weaker bidder drops her demand to zero. So, under clock-truthful bidding, weak bidders would expect to win nothing, therefore submitting a headline demand of $1 - \lambda$ at a clock price of 0 is a profitable deviation.

Remarkably, however, CMRA-truthful bidding turns out to form an ex-post equilibrium in proxy strategies in the CMRA. The reason is, in fact, the same as why a bidder finds it profitable to deviate from clock-truthful bidding. Under CMRA-truthful bidding, the auction ends by the weaker bidder submitting an additional bid of 0 on $1-\lambda$ at clock price p_i^f. As the weaker bidder wins $1-\lambda$ for free, there is no need for her to deviate and demand $1 - \lambda$ at low prices.

Theorem 4. *Suppose that marginal values are non-decreasing (Assumption 2). Then the CMRA has an ex-post equilibrium in proxy strategies in which bidder i bids CMRA-truthfully.*

[10] This is related to what happens in other combinatorial auctions where small bidders need to jointly outbid a larger competitor. In our case, the bidders may wish to coordinate additional bids on small quantities in order to outbid their bids on larger quantities.

Why is CMRA-truthful bidding an equilibrium when bidders have increasing marginal values? The truthful headline demand is λ due to increasing marginal values. Recall that in the clock auction weak bidders have an incentive to deviate to demanding $1-\lambda$ at a clock price of 0 when the other bidder's demand is expected to be λ. Additional bids make such a deviation unnecessary as weak bidders can place an additional bid with value 0 on $1-\lambda$ at high clock prices while keeping the headline demand at λ. Now consider any deviating bid $D_i(x)$ for quantity x. Such a bid must be sufficiently high as it becomes winning only if it is part of a revenue-maximizing allocation. In particular, the deviating bid $D_i(x)$ must be sufficiently high so that

$$D_i(x) + B_j(1-x;p) \geq B_j(\lambda;p).$$

Intuitively, the pay-as-bid pricing rule makes such high deviating bids unattractive.

Remarkably, here the bidders' payments are exactly the same as payments in a VCG auction in which bidders bid truthfully. In such a VCG auction, the strong bidder i wins λ and pays $U_j(\lambda)-U_j(1-\lambda)$, and the weak bidder j wins $1-\lambda$ and pays $U_i(\lambda)-U_i(\lambda)=0$. These are the same payments as in Theorem 4. The equivalence does *not* stem from reporting true marginal values. Instead, the weak bidder's indifference condition at p_j^f plays the key role. Note that the CCA has an equilibrium in truthful strategies that implements the VCG outcome (Levin and Skrzypacz, 2016). Hence, the CMRA and the CCA have outcome-equivalent equilibria under non-decreasing marginal values.

5.2 Equilibrium strategy for any marginal values

We now describe another simple bidding strategy which turns out to be an ex-post equilibrium irrespective of what bidders' marginal values are (as long as they have the same sign). This strategy mimics the essential parts of the CMRA-truthful strategy under non-decreasing marginal values. The bidders submit a headline demand of λ for as long as it gives them non-negative surplus. There are no additional bids except for a single additional bid of 0 on $1-\lambda$ at clock price p_i^f (at which bidder i is indifferent between winning λ and winning $1-\lambda$ for free). If bidder i is stronger, she wins λ at a linear price of p_j^f. If bidder i is weaker, she wins $1-\lambda$ at a price of p_i^f for free. We refer to this strategy as the *constant bidding* strategy. With non-decreasing marginal values, the payments are again the same as in the VCG auction.

Theorem 5. *Suppose that both bidders either have decreasing marginal values (Assumption 1) or non-decreasing marginal values (Assumption 2). Then the CMRA has an* *ex-post equilibrium in proxy strategies in which bidder i's headline demand is*

$$h_i(p) = \begin{cases} \lambda & \text{for } p \leq U_i(\lambda)/\lambda \\ 0 & \text{otherwise;} \end{cases} \tag{3}$$

and the only additional bid made is $A_i(1-\lambda; p_i^f) = 0$. *Moreover, if marginal values are strictly decreasing (Assumption 1), the final allocation is inefficient; and if marginal values are non-decreasing (Assumption 2), the outcome is as under CMRA-truthful bidding.*

Let us confirm that the constant bidding strategy is a best response to itself for any type profile. Let $\theta_i \geq \theta_j$. Then bidder i knows that she wins λ for a price of p_j^f. She cannot win λ at a lower price. Winning λ at p_j^f is better than winning $1-\lambda$ for free as $p_j^f \leq p_i^f$. She could win less than $1-\lambda$ but this is not profitable. Now, if $\theta_i \leq \theta_j$, then bidder i knows she wins $1-\lambda$ for free. She cannot win $1-\lambda$ at a lower price. She cannot win more as this would be more expensive and therefore not profitable.

The argument above does not depend on what the marginal values are (as long as they have the same sign). With non-decreasing marginal values the final allocation and revenue is the same as under CMRA-truthful bidding. With decreasing marginal values, bidders "expand" demand because their headline demands under constant bidding are higher than their headline demands under the clock-truthful bidding. This is a rather surprising equilibrium because one typically associates clock auction and pay-as-bid auction formats with demand reduction rather than demand expansion. Here, the demand expansion leads to an inefficient allocation in which the strong bidder wins an even greater quantity than in an efficient allocation. If there is such bidder asymmetry in the market, one option for the auction designer would be to introduce caps that are tighter on the stronger bidder than on the weaker one.

The constant strategy also provides an opportunity to raise rivals' costs despite the presence of pay-as-bid pricing in the CMRA.[11] Suppose bidder i knows that the other bidder is stronger and bids according to constant strategy. A best response for bidder i is to demand $1-\lambda$ at clock price 0. However, following the constant strategy leads to a higher price for bidder i. In particular, bidder i with a lexicographic preference for raising bidder j's payment will follow the constant strategy; extending the clock beyond p_i^f leads to the risk of winning λ at clock price p_j^f.

[11] It is well-known that VCG pricing can provide ample opportunities to raise rival's cost in the CCA (Janssen and Karamychev, 2016; Levin and Skrzypacz, 2016; Janssen and Kasberger, 2019).

5.3 Riskless demand reduction

We now discuss how additional bids can be used to collude via demand reduction in the CMRA. As we discussed above, bidders have incentives to "reduce demand" in a clock auction and to coordinate in order to close the auction at low prices. However, if a bidder reduces demand, the activity rule (monotonicity constraint) requires that demand stays low as the clock price rises. Therefore, unilateral demand reduction can be a risky strategy.

In the CMRA, in contrast, additional bids can be on smaller quantities while headline demand is high at low clock prices. In particular, at low clock prices, bidders can use additional bids on small quantities to find a revenue-maximizing allocation at which a bid from each bidder is accepted. Demand reduction using additional bids is therefore riskless as neither bidder loses their ability to demand large quantities at higher clock prices if their coordination at low clock prices is unsuccessful.

Concretely, we propose the following collusive strategy in the CMRA. In the first clock round, bidder i submits a headline demand of λ and an additional bid of 0 for quantity 1/2. If both bidders bid this way, the revenue-maximizing allocation immediately assigns half of the supply to each bidder and the revenue is 0. Note that as the headline demand is λ, the bidder can later still demand any quantity if the auction does not end in the first round. This makes the demand reduction strategy riskless for bidders. To state the entire (equilibrium) strategy formally, we modify the constant bidding strategy from the previous section (Theorem 5). The only change is the insertion of the additional bid $A_i(1/2, 0) = 0$. We call this modified strategy the *riskless demand reduction* (RDR) strategy.[12]

For the result, we assume that valuations are quadratic in θ, $\partial^2 U(x;\theta)/\partial\theta^2 = 0$. We make this assumption to simplify the proof as it guarantees that the highest type has the strongest incentive to deviate.[13] For the ease of exposition, we normalize utility levels such that $U_i(\lambda) - U_i(1-\lambda) = \theta_i$.

Assumption 3. Both bidders either have decreasing marginal values (Assumption 1) or non-decreasing marginal values (Assumption 2). Moreover, valuations are quadratic in utility $(\partial^2 U_i(x)/\partial\theta_i^2)$ and $U_i(\lambda) - U_i(1-\lambda) = \theta_i$.

The next theorem shows that both bidders following the riskless demand reduction strategy is a Bayes-Nash equilibrium in proxy strategies if each bidder considers the other bidder to be sufficiently strong in expectation. Note that we now consider Bayes-Nash equilibria in proxy strategies. Riskless demand reduction is not an ex-post equilibrium

[12]The existence of this equilibrium relies on a focal allocation that splits the market and is acceptable to every bidder. In practice, there might be no such focal allocation or indeed many of them (so bidders would need to use multiple additional bids to reach one.)

[13]If the highest type is not the one with the strongest incentive to deviate, then the condition for the existence of the riskless demand reduction equilibrium depends on this type.

as a strong bidder who knows that they are facing a weak bidder has little incentive to collude.

Theorem 6. *Suppose that Assumption 3 holds. Then the profile of riskless demand reduction strategies forms a Bayes-Nash equilibrium in proxy strategies if and only if*

$$\mathbb{E}(\theta_j) \geq U(\lambda;\overline{\theta}) - U(1/2;\overline{\theta}).$$

The proof boils down to checking that winning a quantity 1/2 for free is preferred to winning λ with probability $F(\theta_i)$ for a payment of $\lambda\mathbb{E}_{\theta_j}(p_j^f|\theta_j \leq \theta_i)$ and $1-\lambda$ with probability $1 - F(\theta_i)$ for free. Since the valuations are quadratic in θ, the strongest type $\overline{\theta}$ has the strongest incentive to deviate, yielding the necessary and sufficient condition in the theorem. The normalization of the utility function simplifies the expression for the price p_j^f equal to θ_j/λ. So if the expected payment θ_j under the constant strategy is sufficiently high, there is no incentive to deviate.

We can use Theorem 6 to derive comparative statics results for when collusion is likely to be profitable. First, as $U(\lambda;\overline{\theta}) - U(1/2;\overline{\theta})$ increases, there are fewer type distributions that satisfy the incentive constraint. We can interpret this observation as saying that whenever valuations exhibit stronger increasing returns, riskless demand reduction is less likely to be profitable. For example, consider the parameterized valuation $U(x;\theta) = \theta x^\alpha/(\lambda^\alpha - (1-\lambda)^\alpha)$. As α increases, the valuation function becomes more convex. Moreover, a bidder's value of winning 1/2 becomes lower relative to winning λ. Therefore, a higher expected payment $\mathbb{E}(\theta_j)$ is needed to sustain collusion. Finally, it is worth observing that a tighter cap (i.e., lower λ) makes riskless demand reduction a Bayes-Nash equilibrium in proxy strategies for more type distributions.

6 The CMRA in practice: spectrum auctions in Denmark

In the previous sections we have identified four patterns of bidding: clock-truthful, CMRA-truthful, the constant strategy, and riskless demand reduction. Some of these form equilibria, others do not. In equilibrium there can be demand expansion, demand reduction, and (CMRA-) truthful bidding. We now investigate actual implementations of the CMRA to see what bidding patterns emerged in practice.

The ideal test of bidding behavior would examine actual bidding data. However, the published auction data do not allow us to cleanly identify which pattern of bidding emerged in each auction. The DEA only publishes aggregated payments and allocations for each bidder. We can partially recover whether or not headline or additional bids won.

In our model, each bidding pattern we analyzed has different implications for how the auction ends. If only headline demands are winning, the pattern of bidding is consistent with clock-truthful bidding (or with other clock auction behavior such as standard demand reduction) and we can rule out CMRA-truthful bidding, constant bidding and the riskless demand reduction strategy. If a mix of headline and additional bids is winning, the pattern of bidding is consistent with CMRA-truthful bids and constant bidding and we can rule out clock-truthful bidding and the riskless demand reduction strategy. Finally, if only additional bids are winning, the pattern is consistent with the riskless demand reduction strategy and CMRA-truthful bidding under decreasing marginal values, but we can rule out CMRA-truthful bidding in the case of non-decreasing marginal values, as well as clock-truthful bidding and constant bidding. Of course, we cannot distinguish between CMRA-truthful bidding under non-decreasing marginal values and constant bidding from winning data only because they are outcome equivalent.

We now describe the outcomes of three Danish spectrum auctions that used the CMRA.[14] In each auction there were three bidders whom we refer to as A, B, and C. Each auction allocated lots with and without a coverage obligation. The structure of the auction was such that first the lots with a coverage obligation were allocated (at reserve prices), and then a CMRA was used for the remaining lots. The CMRA first allocated band-specific generic lots. A subsequent assignment phase allocated the specific frequencies.

The outcomes changed as bidders became more acquainted with the auction format. In the first auction, bidders did not use the additional bids to end the auction. In the second auction, there is evidence that additional bids were winning. A bidder may have ended the auction with a low additional bid that was made on a small package. In our theoretical framework, this would amount to the constant strategy. In the third auction, there is again evidence that additional bids were winning. The prevalence of additional bids in later auctions suggests that bidders might have become more familiar with the auction format over time.

6.1 Denmark's 1800 MHz spectrum auction

In 2016, the DEA sold licenses in the 1800 MHz band. The bidders were bidding for 2x65 MHz paired frequencies in the 1800 MHz frequency band. The auction first allocated three 2x10 MHz blocks with a coverage obligation non-competitively: Each bidder won a 2x10 MHz block at the reserve price of DKK 50 million.[15] The remaining 2x35 MHz were sold in a CMRA. After assigning the blocks with the coverage obligation, there were

[14]We do not consider the 2020 Norwegian spectrum CMRA. As this auction allocated licenses in an unusually large number of different frequency bands, it is probable that the main task of the auction was achieving coordination among the bidders rather than creating competition for scarce licenses. Excess supply in the auction is evidence for this.

[15]On 25 February 2022, DKK 1 = $0.15.

spectrum caps that allowed each bidder to win at most 2x20 MHz in the CMRA (which corresponds to a λ of 0.57).

The CMRA allocated seven lots (2x5 MHz blocks). Each bidder was allowed to win at most four blocks in the CMRA. The reserve price was DKK 25 million. The CMRA allocated generic blocks; specific frequencies were allocated in an assignment stage after the CMRA. The assignment stage used a VCG auction.

Bidder A won 2x20 MHz for a total price of DKK 300,159,486. Bidder B won 2x20 MHz for a total price of DKK 300,159,486. Bidder C won 2x25 MHz for a total of DKK 425,239,229. Each bidder won 2x10 MHz with the coverage obligation, so bidders A and B both won two lots in the CMRA. Bidder C won three lots in the CMRA.

It is likely that only headline demands were winning. The first evidence is that bidders A and B paid exactly the same amounts. Moreover, after subtracting the reserve price for the 2x10 MHz with the coverage obligation, the payment per lot in the CMRA was exactly DKK 125,079,743 *for each bidder*. We view this as evidence that the final clock price was DKK 125,079,743. It also suggests that the VCG prices in the assignment stage were 0.

The CMRA ending with headline demand only is only consistent with clock-truthful bidding. While it is probable that bidders did not submit truthful headline demands but reduced their headline demands, there is no evidence that they used additional bids to end the auction as they would in CMRA-truthful bidding, in the constant strategy, or in riskless demand reduction.

6.2 Denmark's 700 MHz, 900 MHz and 2300 MHz auction

In early 2019, the DEA auctioned licenses for frequencies in the 700 MHz, 900 MHz, and 2.3 GHz bands. The DEA used the CMRA to sell 16 blocks across two bands. Revenue was almost DKK 2.2 billion. A 40 MHz block in the 2.3 GHz band was unsold. We believe that it did not sell due to an associated coverage obligation.

There was a large difference in bidders' payments. In the Appendix B.1, we argue it is plausible that bidder B won their headline demand, bidder C won with an additional bid, and bidder A either won their headline demand or with an additional bid. Bidder B won the maximum quantity allowed by the spectrum cap (excluding the unsold lot with the coverage obligation that received no bids), which is consistent in our model with CMRA-truthful bidding under increasing marginal values or with the constant strategy. Bidder C won a small quantity at a relatively low price, which is also consistent with these two strategies.

6.3 Denmark's 2021 auction

In 2021, the DEA allocated frequencies in the 1.5 GHz, 2.1 GHz, 2.3 GHz, 3.5 GHz and 26 GHz bands. Again, the DEA first auctioned off blocks with coverage obligations and then used the CMRA to auction the rest. The DEA used the CMRA to sell a total of 39 blocks across all the five bands. There were three bidders and revenue was just above DKK 2 billion. The auction ended with no lots unsold. If all lots had sold at reserve price, revenue would have been DKK 865 million. In Appendix B.2, we argue that at least some additional bids were winning as linear closing prices seem unlikely. Again, this suggests that bidders were not simply using headline demands.

6.4 What theory predicts about bidding dynamics

Some telecom regulators are reluctant to publish bidding data in spectrum auctions because it can contain highly sensitive commercial information. However, our theory makes predictions about the possible dynamics of bidding which could allow regulators to assess whether observed bids are consistent with the patterns we describe. In particular, bidding data would allow an observer to distinguish between CMRA-truthful bidding with decreasing marginal values vs. riskless demand reduction strategy and between CMRA-truthful bidding with non-decreasing marginal values vs. the constant strategy. We cannot distinguish between these from outcome data alone. Under CMRA-truthful bidding, a bidder starts with none or few additional bids. In the course of the auction, they not only add new additional bids but also raise existing additional bids. Under the riskless demand reduction strategy, a bidder starts with a headline demand of a large quantity and many additional bids for smaller quantities.[16] This cannot happen under truthful bidding because a bidder cannot be indifferent between a large and small quantity of spectrum at equally low prices. Under riskless demand reduction, we therefore expect high variance in the bid quantities in early rounds and for the auction to end quickly. Under the constant bidding strategy, the headline demand is constant and additional bids only come later in the auction. The additional bid in this case is low but is made at a relatively high clock price (i.e, at bidder i adds it at clock price p_i^f). Fig. 5 summarizes which bidding patterns can be explained by our results and suggests which bidding patterns in the CMRA would require new theoretical results.

[16] In our model, bidders only submit one additional bid in the riskless collusive strategy, but in practice with many lots across many bands, it might make sense to submit a number of additional bids which would reflect salient ways of splitting the market. Indeed, bidders have many opportunities to coordinate on a collusive market split at clock prices higher than the reserve.

Winning bids	Additional bids in bidding data: Few early, more later	Many early	None or few
Headline only	NA	NA	Clock-truthful
Headline/additional	CMRA-truthful (non-dec.)	NA	Constant bidding
Additional only	CMRA-truthful (dec.)	Riskless dem. red.	NA

Figure 5: Predicted bidding patterns under different bidding strategies and marginal valuations.
Note: "NA" are bidding patterns that cannot be explained by our results.

7 Conclusion

In this paper, we provided the first theoretical analysis of the CMRA, an auction format that has been used in several European spectrum sales. Our results suggest that this new auction format should perform well compared to the clock auction when the goods for sale are complements. However, the CMRA is potentially prone to riskless collusive bidding when goods are not sufficiently complementary. In spectrum auctions, there are often substitutes and complements for sale, so the auction designer would need to (and they typically do) carefully weigh up the pros and cons of different auction formats.

There are many avenues for further work. First, one could develop the theoretical analysis by looking at other equilibria in our model, by introducing more asymmetry (e.g., one bidder with decreasing marginal values and a bidder with increasing marginal values), or by extending the model to heterogeneous goods. Second, one could take our theoretical predictions to real-world or experimental bidding data.[17] Third, one could analyze bidder incentives under richer utility functions that include, for example, spitefulness or risk aversion.

[17] The CCA has been extensively studied in experimental settings. See, for example, Kagel et al. (2010) and Kagel et al. (2014).

A Omitted Proofs

A.1 Proofs in Section 4

Proof of Theorem 1. We prove each of the three statements separately.

1. The truthful headline demands are continuous and decreasing in p. Thus, the auction ends with market clearing at price $p^\star$. Truthful bidding implies that it ends in the efficient allocation.

2. We first show that CMRA-truthful bidding cannot end in an inefficient allocation. Let x be an inefficient allocation in which bidder i wins, wlog, less than in the efficient allocation ($x_i < x_i^\star$). The allocation x can only be implemented if both bidders have bid their respective shares. Then bidder i's marginal bid is higher than bidder j's marginal bid. Thus it is profitable to allocate more to bidder i.

   In any round p, the revenue-maximizing allocation is either the efficient allocation or a boundary allocation ($(\lambda, 0)$ for low prices or $(\lambda, 1-\lambda)$ for high prices). Note that $1-\lambda < x_i^\star$. Hence, there are prices p for which bidder i bids on $x_i^\star$ but not on $1-\lambda$. For prices p for which at least one bidder bids on $1-\lambda$, we have

   $$B_i(x_i^\star; p) + B_j(x_j^\star; p) = U_i(x_i^\star) - V_i(p) + U_j(x_j^\star) - V_j(p) > B_i(\lambda; p) + B_j(1-\lambda; p).$$

   It follows that the CMRA does not implement $(\lambda, 1-\lambda)$.

   The CMRA then ends at the lowest price $\tilde{p}^\star$ such that

   $$B_i(x_i^\star; \tilde{p}^\star) + B_j(x_j^\star; \tilde{p}^\star) = \max_{i=1,2} B_i(\lambda; \tilde{p}^\star);$$

   the auction ends as soon as the interior (efficient) revenue-maximizing allocation is a global maximum of revenue. We have $\tilde{p}^\star < p^\star$ as

   $$B_i(x_i^\star; p^\star) + B_j(x_j^\star; p^\star) = p^\star > p^\star\lambda > B_i(\lambda; p^\star) = U_i(\lambda) - U_i(x_i^\star) + p^\star x_i^\star$$

   due to decreasing marginal values (bids) and truthful headline demands. The inequality implies that at $p^\star$ the unique revenue-maximizing allocation is the efficient allocation, so the auction must have ended at a lower price.

3. We now show that revenue is lower. The CMRA ends at $\tilde{p}^\star < p^\star$. Bidder i's contribution to revenue is

   $$B_i(x^\star; \tilde{p}^\star) = U_i(x^\star) - V(\tilde{p}^\star) \leq x^\star\tilde{p}^\star < x^\star p^\star.$$

The first inequality is true as the efficient share is not the utility maximizing demand at $\tilde{p}^\star$.

□

Proof of Theorem 2. We prove each of the three statements separately.

1. The clock-truthful demand is λ until the price reaches $U_i(\lambda)/\lambda$. For higher prices demand equals 0. Hence, the clock ends with excess supply. Due to positive marginal values for all shares below λ, the efficient allocation does not feature excess supply. The outcome is inefficient.

2. Consider CMRA-truthful bidding. For $p \leq U_i(\lambda)/\lambda$, truthful additional bids are $B_i(x;p) = U_i(x) - V_i(p)$, where $V_i(p) = U_i(\lambda) - p\lambda$. At price p_i^f bidder i is indifferent between winning λ for a payment λp_i^f and winning $1-\lambda$ for free. Hence, the auction ends at clock price $\min\{p_1^f, p_2^f\}$ with market clearing. Note that $p_i^f < U_i(\lambda)/\lambda$, where $U_i(\lambda)/\lambda$ is the price at which bidder i drops out by bidding 0 on 0. The inequality follows immediately from the respective indifference conditions. Hence, the auction ends by one bidder submitting a 0 bid on $1-\lambda$ and not by one bidder dropping out.

   We now show that the weak bidder j first bids 0 on $1-\lambda$. To do so, note that

   $$\frac{\partial p_i^f}{\partial \theta_i} = \frac{1}{\lambda} \frac{\partial \int_{1-\lambda}^{\lambda} u_i(x)dx}{\partial \theta_i} = \frac{1}{\lambda} \int_{1-\lambda}^{\lambda} \frac{\partial^2 U_i(x)}{\partial \theta_i \partial x} dx.$$

   The first derivative is positive as marginal values increase in θ_i. Hence, p_i^f is lower for the lower type. As the lower type bids earlier 0 on $1-\lambda$, the auction ends in the efficient allocation.

3. Under clock-truthful bidding, ex-post revenue is $\lambda \min_i U_i(\lambda)/\lambda$. Under CMRA-truthful bidding the auction ends at a lower price p^f. Ex-post revenue is $\lambda p^f + 0$.

□

A.2 Proofs in Section 5

Proof of Theorem 3. Suppose both bidders bid CMRA-truthfully. The auction ends at price $\tilde{p}^\star$ such that

$$B_1(x_1^\star; \tilde{p}^\star) + B_2(x_2^\star; \tilde{p}^\star) = \max_i B_i(\lambda; \tilde{p}^\star).$$

For almost all values, there is a positive probability that bidder i has a higher bid on λ. Hence, bidder i has an incentive to stop raising bids on quantities that can no longer be efficient shares. □

Proof of Theorem 4. Suppose bidder j bids CMRA-truthfully. If bidder i also bids in such a way and $\theta_i \geq \theta_j$, then the auction ends with bidder i winning λ for a payment $p_j^f \lambda$. If $\theta_i < \theta_j$, then bidder i wins $1-\lambda$ for free. In the first case the surplus is $U_i(\lambda) - U_j(\lambda) + U_j(1-\lambda)$. In the second case the surplus is $U_i(1-\lambda)$.

Suppose bidder i wants to win x for a payment $D_i(x)$. As bidder j's headline demand is λ, bidder i's bid $D_i(x)$ at price p is winning only if

$$D_i(x) + B_j(1-x;p) \geq B_j(\lambda;p).$$

Ideally, this inequality is binding as otherwise the bid $D_i(x)$ can be decreased. Plugging in the expressions for bidder j's bids yields

$$D_i(x) = U_j(\lambda) - U_j(1-x).$$

Consider $\theta_i \geq \theta_j$. The deviation is profitable only if the surplus from winning x is better than winning λ, i.e.,

$$U_i(x) - D_i(x) \geq U_i(\lambda) - p_j^f \lambda.$$

Plugging in the expression for $D_i(x)$ gives

$$U_i(x) - U_j(\lambda) + U_j(1-x) \geq U_i(\lambda) - U_j(\lambda) + U_j(1-\lambda).$$

Note that this is a (weak) contradiction as it is efficient that bidder i wins λ. We conclude that bidder i does not have an incentive to deviate.

Finally, consider $\theta_i < \theta_j$. The deviation is profitable if

$$U_i(x) - U_j(\lambda) + U_j(1-x) \geq U_i(1-\lambda),$$

where the left-hand side is the expected utility of winning x for a payment $D_i(x)$ and the right-hand side is the expected utility from following the CMRA-truthful strategy. The inequality holds as bidder j winning λ is efficient. □

Proof of Theorem 5. The proof follows similar steps as the proof of Theorem 4. Suppose both bidders follow the constant strategy. If $\theta_i \geq \theta_j$, then the auction ends with bidder i winning λ for payment $p_j^f \lambda$. If $\theta_i < \theta_j$, then bidder i wins $1-\lambda$ for free. In the first case the surplus is given by Eq. (A.2). In the second case the surplus is $U_i(1-\lambda)$.

Suppose that instead of following the constant strategy, bidder i wants to win x_i for bid $D_i(x_i)$ at clock price p. As bidder j's headline demand is λ for any clock price below $U_j(\lambda)/\lambda$, bidder j's bid on λ is $B_j(\lambda;p) = \min\{p, U_j(\lambda)/\lambda\}\lambda$. Importantly, to be part of the revenue-maximizing allocation, the bid $D_i(x_i)$ must satisfy

$$D_i(x_i) + B_j(x_j; p) \geq B_j(\lambda; p),$$

where $x_i + x_j \leq 1$.

Consider $p < p_j^f$. As bidder j's headline demand is λ and bidder j places no additional bids, bidder i faces

$$B_j(x_j; p) = \begin{cases} -\infty & \text{for } 0 \leq x_j < \lambda \\ p\lambda & \text{for } x = \lambda. \end{cases}$$

Hence, the only feasible x_i is $1-\lambda$. If $\theta_i \geq \theta_j$, then $1-\lambda$ can be won for free at p_j^f (which is at least as good as winning $1-\lambda$ with bid D_i). If $\theta_i < \theta_j$, then bidder i cannot do better by deviating from the constant strategy.

Consider $p \in [p_j^f, U_j(\lambda)/\lambda]$. At the clock price p_j^f bidder j adds the additional bid on $1-\lambda$, which changes the bid function at $1-\lambda$ to $B_j(1-\lambda; p) = 0$. Now the revenue-maximizing allocation is $(x_i, 1-\lambda)$ if $D_i(x_i) \geq B_j(\lambda; p)$. In the best case the inequality is binding. Hence, if $D_i(x_i)$ is winning, then bidder i's utility at most $U_i(x_i) - B_j(\lambda; p)$. The x_i that maximizes the utility is $x_i = \lambda$. If $\theta_i \geq \theta_j$, then bidder i wins λ for $p_j^f \lambda$, which is at least as good as winning λ for price $B_j(\lambda; p) = p\lambda \geq p_j^f \lambda$. Hence, there is no profitable deviation when bidder i is stronger. If bidder i is weaker, then the utility of winning λ for $B_j(\lambda; p)$ is less than winning $1-\lambda$ for free, which can be achieved by the constant strategy. Hence, there is also no profitable deviation if bidder i is weaker than bidder j.

Finally, consider $p > U_j(\lambda)/\lambda$. Bidder j bids

$$B_j(x_j; p) = \begin{cases} 0 & \text{for } x_j \in \{0, 1-\lambda\} \\ U_j(\lambda) & \text{for } x = \lambda \\ -\infty & \text{else.} \end{cases}$$

Winning x_i now becomes more expensive compared to the previous case. There is no profitable deviation to the constant strategy. □

Proof of Theorem 6. We first characterize the payoff if a bidder deviates. If a bidder deviates, then the outcome is as in the ex-post equilibrium described in Theorem 5, i.e., the bidder with the higher type θ wins the entire supply. Bidder j demands λ for p such that $U_i(\lambda) - p\lambda \geq 0$. At $p_j^f = \frac{1}{\lambda}\left(U_j(\lambda) - U_j(1-\lambda)\right)) = \frac{\theta_j}{\lambda}$ the additional bid on $1-\lambda$ is made. Thus, if $\theta_i \geq \theta_j$, the auction ends at p_j^f. The expected utility of bidder i is then

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta_i} U_i(\lambda) - p_j^f \lambda \, dF(\theta_j) + (1 - F(\theta_i))U_i(1-\lambda) \\
&\qquad\qquad = \int_{\underline{\theta}}^{\theta_i} U_i(\lambda) - \theta_j \, dF(\theta_j) + (1 - F(\theta_i))U_i(1-\lambda) \\
&\qquad\qquad = F(\theta_i)(U_i(\lambda) - U_i(1-\lambda) + U_i(1-\lambda) - \int_{\underline{\theta}}^{\theta_i} \theta_j f(\theta_j) \, d\theta_j \\
&\qquad\qquad = F(\theta_i)\theta_i + U_i(1-\lambda) - \int_{\underline{\theta}}^{\theta_i} \theta_j f(\theta_j) \, d\theta_j.
\end{aligned}$$

We verify the following incentive constraint that RDR leads to higher expected utility than deviating. The incentive constraint requires

$$U_i(1/2) - \left(F(\theta_i)\theta_i + U_i(1-\lambda) - \int_{\underline{\theta}}^{\theta_i} \theta_j f(\theta_j) \, d\theta_j \right) \geq 0. \tag{IC}$$

We show that the inequality holds for all θ_i. To do so, we minimize the left-hand side. The first derivative of the left-hand side with respect to θ_i is

$$\frac{\partial U(1/2;\theta_i)}{\partial \theta_i} - \left(f(\theta_i)\theta_i + F(\theta_i) + \frac{\partial U(1-\lambda;\theta_i)}{\partial \theta_i} - \theta_i f(\theta_i) \right).$$

The second derivative with respect to θ_i is

$$\frac{\partial^2 U(1/2;\theta_i)}{\partial \theta_i^2} - \frac{\partial^2 U(1-\lambda;\theta_i)}{\partial \theta_i^2} - f(\theta_i).$$

The assumption that valuations are quadratic in θ implies that the second derivative is negative. A concave function is minimized on the boundary. Thus, if (IC) holds for $\overline{\theta}$, it holds for all θ_i. Plugging $\overline{\theta}$ into (IC) leads to the stated condition on the expected type. □

B Further details on the Danish spectrum auctions

B.1 Details on the 2019 Danish auction

Table 1 summarizes the supply in the 2019 auction and the auction outcome. The licenses in the 900 MHz band were not allocated through a CMRA. In the 900 MHz band, there were 2x30 MHz paired frequencies available. These licenses came with a coverage obligation. We call the lots in the 900 MHz band A lots.

The supply in the CMRA was six 2x5 MHz blocks (B lots) of paired frequencies in the 700 MHz band, four 5 MHz blocks of unpaired frequencies in the 700 MHz band (D lots), one block 40 MHz block in the 2.3 GHz band with a coverage obligation (E lot), and six 10 MHz blocks in the 2.3 GHz band (F lots).

There was no reserve price on the lots with a coverage obligation. The reserve price per B lot was DKK 95 million. The reserve price per D lot was DKK 25 million. The reserve price per F lot was DKK 25 million.

The following spectrum caps were in place. Each bidder was allowed to win at most one block in the 900 MHz band. Across the paired blocks in the 700 MHz and 900 MHz bands, each bidder was allowed to win at most four lots. Bidders were not allowed to win more than 60 MHz in the 2.3 GHz band. There was no restriction on the number of blocks a bidder could win in the unpaired 700 MHz band.

The auction outcome was as follows. Bidder A paid DKK 485.2 million for one A lot and two B lots. Bidder B paid DKK 1620 million for one A lot, three B lots, four D lots, and six F lots. Bidder C paid DKK 107.6 million for one A lot and one B lot. Hence, the 40 MHz lot in the 2.3 GHz spectrum with the coverage obligation was unsold. Note that bidder B received the maximum quantity permitted by the spectrum caps, which is consistent with the CMRA-truthful and constant strategies. For this reason, it is likely that bidder B won with their headline demand.

We now examine whether the bidders paid linear prices. Recall this would suggest that only headline demands were winning. As the A lots were traded at a reserve price of 0, bidder C paid DKK 107.6 million for a single B lot. Bidder A paid DKK 485.2 million for two B lots. As bidder A paid more than four times bidder C's payment, we take this as evidence that at least one additional bid was winning. In particular, we speculate that bidder C won with an additional bid. Winning a small package at low cost is consistent with the constant strategy and with CMRA-truthful bidding.

While we think it is likely that bidder B won with a headline demand, it is not clear whether bidder A won with a headline or an additional bid. Bidder B paid DKK 1135 million more than bidder A for also winning another B lot, the four D lots, and the six F lots. If bidder B won the headline demand, then $1135 = p_B + 4p_D + 6p_F$. Suppose bidder A won the headline demand, implying $p_B = 485.2/2 = 242.6$ and $892.4 = 4p_D + 6p_F$.

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
B	2x5 MHz in the 700 MHz band	6	95	2	3	1
D	5 MHz in the 700 MHz band	4	25		4	
E	40 MHz in the 2.3 GHz band	1	0			
F	10 MHz in the 2.3 GHz band	6	25		6	
Non-competitive						
A	2x10 MHz in the 900 MHz band	3	0	1	1	1
Expenditure						
	In million DKK		820	485	1620	107.6

Table 1: Supply and auction outcome in the 2019 Danish spectrum auction
Notes: R = reserve price in million DKK; the E lot came with a coverage obligation

Moreover, assume that $p_D = p_F$ as the D and F lots have the same reserve price. Linear prices then imply that $p_D = p_F = 89.24$, which is not implausible. Conversely, if bidder B won their headline demand and final prices for D and F were, say, about 65, then this would imply $p_B = 485$. In particular, bidder A would have bought two B lots with an additional bid at half price. Hence, we cannot rule out the possibilities that bidder A won their headline demand or with an additional bid. Finally, we do not see any signs that the auctioned ended as in the riskless demand reduction equilibrium.

B.2 Details on the 2021 Danish auction

The process was similar to the two previous auctions. Bidders first had the chance to obtain 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0 (2.1-D lot). All bidders bought such a license. Table 2 summarizes the supply and outcome.

There were two subsequent CMRAs. In the first CMRA, there were ten lots in the 1500 MHz band available: a single 25 MHz (lot 1.5-B) for a reserve price of DKK 10 million, eight 5 MHz lot (1.5-M) for a reserve price of DKK 10 million each, and another single 25 MHz block (1.5-T) for a reserve price of DKK 10 million. In the 2.1 GHz spectrum, there were six 2x5 MHz blocks (2.1-U) available for a reserve price of DDK 25 million each. In the 2.3 GHz band, there were two lots for 20 MHz available for a reserve price of DDK 25 million. In the 3.5 GHz band there were three categories of lots. First, there were three lots in the 3.5 GHz band available (3.5-D). The reserve price for such a lot was DDK 75 million. One such lot corresponds to 80 MHz in the 3.5 GHz spectrum and 400 MHz in the 26 GHz spectrum. Second, there was a single lot of 60 MHz (3.5-P) for a reserve price of DDK 25 million in the 3.5 GHz band with a leasing obligation. Third, there were nine 10 MHz lots (3.5-U) for a reserve price of DDK 25 million. The second CMRA was for the remaining lots in the 26 GHz band. In total there were 2850 MHz unpaired frequencies in the 26 GHz band. After subtracting the 1200 MHz sold in the first auction through the 3.5-D lots, there were 1650 MHz available (in lots of 200 MHz and 250 MHz) in the second CMRA. The reserve price was about DKK 5 million per lot.

Each of the three bidders won a 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0, two 2x5 MHz lots in the 2.1 GHz band, and a 3.5-D lot (80 MHz in the 3.5 GHz band and 400 MHz in the 26 GHz band).

In addition, bidder A won 40 MHz in the 3.5 GHz band (four 3.5-U lots) in the first CMRA and 600 MHz in the 26 GHz band in the second CMRA. Bidder A's total payment was DKK 540,525,000.

In addition to the above, bidder B won the 1.5-B lot, four 1.5-M lots, the two lots in the 2.3 GHz band, and 50 MHz in the 3.5 GHz band (five 3.5-U lots). In the second CMRA, bidder B won 850 MHz in the 26 GHz band. Bidder B's total payment was DKK 794,685,000.

In addition to the above, bidder C won the 1.5-T lot, four 1.5-M lots, and the 60 MHz in the 3.5 GHz spectrum with the leasing obligation. In the second CMRA, bidder C won 200 MHz in the 26 GHz band. Bidder C's total payment was DKK 740,976,000.

We first look at the differences between bidders B and C. Bidder B won the two 2.3-U lots in the 2.3 GHz band (40 MHz in total), 10 MHz less in the 3.5 GHz band (but without the leasing obligation), and 650 MHz more in the 26 GHz band. Bidder B paid DKK 53,709,000 more than bidder C. Bidder B's final assignment seems to dominate bidder C's and cost only DKK 54 million more. Compare this number to the reserve price of DKK 100 million for the 2.3 GHz band alone. Hence, we suspect that bidder B used additional bids to win the large package (as under CMRA-truthful bidding with decreasing marginal values).

Next, we compare the outcomes of bidders A and B. Bidder B paid DKK 254 million more than bidder A and got the additional 45 MHz in the 1500 MHz band, 40 MHz in the 2300 MHz band, 10 MHz in the 3.5 GHz band, and 250 MHz in the 26 GHz band. The reserve price for the additional lots won by bidder B is DKK 180 million. Hence, bidder B paid DKK 74 million in excess of the reserve price.

Comparing bidders A and C, bidder C won 45 MHz in the 1500 MHz band while bidder A did not win any lot in this category. Bidder C won 20 MHz more in the 3.5 GHz band (but subject to the leasing obligation), and 400 MHz less in the 26 GHz band. Bidder A paid DKK 200 million less, however. The reserve price of the 45 MHz in the 1500 MHz band was DDK 50 million.

We conclude that it is likely that bidder B won with an additional bid. Due to the many prices, we cannot say whether bidders A and B won their headline demands or with additional bids. There is, however, no evidence for riskless demand reduction.

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
1.5-B	25 MHz in the 1500 MHz band (bottom)	1	10		1	
1.5-M	5 MHz in the 1500 MHz band	8	10		4	4
1.5-T	25 MHz in the 1500 MHz band (top)	1	10			1
2.1-U	2x5 MHz in the 2.1 GHz band	6	25	2	2	2
2.3-U	20 MHz in the 2.3 GHz band	2	50		2	
3.5-D	80 MHz in 3.5 GHz + 400 MHz in 26 GHz	3	75	1	1	1
3.5-P	60 MHz in 3.5 GHz (leasing obligation)	1	25			1
3.5-U	10 MHz in the 3.5 GHz band	9	25	4	5	
26-U	200 MHz/250 MHz in the 26 GHz band	8	5	3	4	1
Non-competitive						
2.1-D	2x10 MHz in the 2.1 GHz band	3	0	1	1	1
Expenditure						
	In million DKK		865	541	795	741

Table 2: Supply and auction outcome in the 2021 Danish spectrum auction
Note: R = reserve price in million DKK

References

Ausubel, L. M., P. Cramton, and P. Milgrom (2006). The clock-proxy auction: A practical combinatorial auction design. In M. Bichler and J. K. Goeree (Eds.), *Handbook of Spectrum Auction Design*, pp. 120–140. Cambridge, MA: MIT.

Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014). Demand reduction and inefficiency in multi-unit auctions. *Review of Economic Studies 81*(4), 1366–1400.

Ausubel, L. M. and P. R. Milgrom (2002). Ascending auctions with package bidding. *Advances in Theoretical Economics 1*(1), 20011001.

Baranov, O. (2018). An Efficient Ascending Auction for Private Valuations. *Journal of Economic Theory 177*, 495–517.

Baranov, O., C. Aperjis, L. M. Ausubel, and T. Morrill (2017). Efficient procurement auctions with increasing returns. *American Economic Journal: Microeconomics 9*(3), 1–27.

Bichler, M., P. Shabalin, and G. Ziegler (2013). Efficiency with linear prices? a game-theoretical and computational analysis of the combinatorial clock auction. *Information Systems Research 24*(2), 394–417.

Brusco, S. and G. Lopomo (2002). Collusion via Signalling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without Complementarities. *Review of Economic Studies 69*(2), 407–436.

DotEcon (2016, February). The combinatorial multi-round ascending auction (cmra): proposal for a new auction format. Technical report, DotEcon Perspectives.

Grimm, V., F. Riedel, and E. Wolfstetter (2003). Low price equilibrium in multi-unit auctions: the gsm spectrum auction in germany. *International Journal of Industrial Organization 21*(10), 1557–1569.

Janssen, M. and V. Karamychev (2016). Spiteful bidding and gaming in combinatorial clock auctions. *Games and Economic Behavior 100*, 186–207.

Janssen, M., V. Karamychev, and B. Kasberger (2017). Budget constraints in combinatorial clock auctions. In M. Bichler and J. K. Goeree (Eds.), *Handbook of Spectrum Auction Design*, pp. 318–337. Cambridge, England: Cambridge University Press.

Janssen, M. and B. Kasberger (2019). On the clock of the combinatorial clock auction. *Theoretical Economics 14*(4), 1271–1307.

Kagel, J. H., Y. Lien, and P. Milgrom (2010). Ascending prices and package bidding: A theoretical and experimental analysis. *American Economic Journal: Microeconomics 2*(3), 160–85.

Kagel, J. H., Y. Lien, and P. Milgrom (2014). Ascending prices and package bidding: Further experimental analysis. *Games and economic behavior 85*, 210–231.

Kelso Jr, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica 6*(50), 1483–1504.

Levin, J. and A. Skrzypacz (2016). Properties of the combinatorial clock auction. *American Economic Review 106*(9), 2528–51.

Maldoom, D. (2007). Winner determination and second pricing algorithms for combinatorial clock auctions. *DotEcon Discussion Paper*.

Milgrom, P. (2000). Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy 108*(2), 245–272.

Palacios-Huerta, I., D. C. Parkes, and R. Steinberg (2021). Combinatorial auctions in practice. Technical report, Available at SSRN 3844338.

Parkes, D. C. and L. H. Ungar (2002). An ascending-price generalized vickrey auction. Technical report, Paper presented at the Stanford Institute for Theoretical Economics: The Economics of the Internet.

Porter, D., S. Rassenti, A. Roopnarine, and V. Smith (2003). Combinatorial auction design. *Proceedings of the National Academy of Sciences 100*(19), 11153–11157.

Rassenti, S. J., V. L. Smith, and R. L. Bulfin (1982). A combinatorial auction mechanism for airport time slot allocation. *Bell Journal of Economics 13*(2), 402–417.

Sun, N. and Z. Yang (2009). A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica 77*(3), 933–952.

Sun, N. and Z. Yang (2014). An efficient and incentive compatible dynamic auction for multiple complements. *Journal of Political Economy 122*(2), 422–466.

Teytelboym, A. (2014). Gross substitutes and complements: A simple generalization. *Economics Letters 123*(2), 135–138.